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Tensor abelian categories

- in a non-commutative setting

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Abstract

Tensor abelian categories provide a framework for studying both the additive (abelian) and the multiplicative (monoidal) structure of categories like abelian grups, modules over rings, chain complexes, (differential) graded modules, quasi-coherent sheaves and functor categories, even in the non-commutative setting. In the first paper, we prove in this framework a classic theorem of Lazard and Govorov which states that flat modules are precisely the direct limit closure of the finitely generated projective modules. The general result reproves this and other ad hoc examples and provide new results in other categories including the category of differential graded modules. In the second paper we study quiver representations in such categories and characterize various classes of representations. This again generalizes characterizations in R-Mod, but provides new insight even in this case. In the last paper we study a generalization of the prime ideal spectrum in this setting, namely the atom spectrum. This has many good theoretical properties but concrete calculations are few. We provide a method for calculating this with several concrete examples.

Resumé

Tensor-abelske kategorier er en teori hvormed man kan studere både de additive og multiplikative strukture af kategorier såsom kategorien af abelske grupper, moduler over ringe, kædekomplekse, (differential) graduerede ringe, kvasi-koherente knipper og funktorkategorier selv i det ikke-kommutative tilfælde. I det første arbejde løfter vi et klasisk resultat af Lazard og Govorov for moduler over ringe til denne ramme. Dette generelle resultat specialiserer til flere lignende velkendte resultater og giver nye resultater om eksempelvis differential graduerede ringe. I det andet arbejde beskriver vi forskellige klasser af kogger-representationer i disse kategorier. Dette generaliserer beskrivelser af representationer i kategorien af moduler men giver ny indsigt også i dette tilfælde. I det tredje arbejde kikker vi på en generalisering af primeidealspektret til denne ramme kaldet atomspektret. Det har mange fine teoretiske egenskaber men der har hidtil ikke været mange konkrete udregninger. Vi giver en beregningemetode og kommer med flere konkrete udregninger.

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This thesis consitsts of three papers

[I] Dualizable and semi-flat objects in abstract module categories, Math. Z. (to appear), arXiv:1607.02609.

An abstract version of the classic Lazard-Govorov theorem with new applications.

[II] Direct limit closure of induced quiver representations, preprint (2018), arXiv:1805.04169.

General description of the classes $\Phi(\mathscr{X})$ *and* $\Phi(\varinjlim \mathscr{X})$ *of quiver representations with application to Gorenstein homological algebra.*

[III] Computations of atom spectra, preprint (2018), arXiv:1805.04315.

A method for computing atom spectra of Grothendieck categories based on tilings.

I am the sole author of the first two papers. The third is joint work with Henrik Holm. The first paper is accepted for publication in Mathematische Zeitschrift. The last two has been submitted for publication.

Before introducing the content of the three papers we give an introduction to the theory of $\otimes$-abelian categories in a non-commutative setting. We then explain the results of the three papers in view of this theory.

1 Tensor abelian categories

In abstract algebra a central object of study is that of an abelian category. This includes the category of abelian groups, Ab, and the category of R-modules, R-Mod for any ring R. But it also includes categories like chain complexes of R-modules, Ch(R-Mod), (differential) graded R modules R-GrMod, (R-DGMod) where R is a (differential) graded ring, the category $\mathrm{QCoh}(X)$ of quasi coherent sheaves over a scheme X, and functor categories $\mathrm{Fun}(\mathscr{A}, \mathrm{Ab})$ where $\mathscr{A}$ is a small category. The notion of an abelian category captures in some sense the additive structure of the category. If we wish to study the multiplicative structure we have have the notion of monoidal categories. But $\mathrm{Fun}(\mathscr{A}, Ab)$ and R-Mod, Ch(R-Mod) and R-DGMod, when R is not commutative, are not monoidal categories. We do have a $\otimes$-product however, and interesting questions arise when we look at the interplay between this and the additive structure. In this thesis we develop an abstract theory of $\otimes$-abelian categories which include the non-commutative examples above.

1.1 Additive structure

In an abelian category we have the following notion of an object being small:

Definition 1 (Breitsprecher [4, Def. 1.1])**.** An object S in an category A is said to be finitely presented (FP_1) if for every directed system $\{X_i\}$ every map $S \to \varinjlim X_i$ factors through X_i for some i. Another way of saying this is that the canonical map

$$\varinjlim \mathrm{Hom}(S, X_i) \to \mathrm{Hom}(S, \varinjlim X_i)$$

is an isomorphism.

We are interested in categories built out of small objects in the following way.

Definition 2 (Crawley-Boevey [6, §1])**.** An abelian category, $\mathscr{A}$, is locally finitely presented if $FP_1(\mathscr{A})$ is a set, and $\varinjlim FP_1(\mathscr{A}) = \mathscr{A}$, i.e. for every $A \in \mathscr{A}$ there is a directed system $\{X_i\} \subseteq \mathrm{FP}_1(\mathscr{A})$ s.t. $\varinjlim X_i = A$.

The finitely presented objects are part of a tower of small objects.

$$FP_0 \subseteq FP_{0.5} \subseteq FP_1 \subseteq FP_{1.5} \subseteq FP_2 \subseteq \cdots \subseteq FP_n \subseteq FP_{n.5} \subseteq FP_{n+1} \subseteq \cdots$$

where an object is FP_n ($n \geq 1$) if the canonical map

$$\varinjlim \mathrm{Ext}^k(X, Y_i) \to \mathrm{Ext}^k(X, \varinjlim Y_i)$$

is an isomorphism for every $k < n$. It is $FP_{n.5}$ if further the map is monic for $k = n$. We have $FP_0(\mathscr{A}) = FP_{0.5}(\mathscr{A})$ by definition and $FP_1(\mathscr{A}) = FP_{1.5}(\mathscr{A})$ by Stenström [28, Prop. 2.1] when $\mathscr{A}$ is AB5 (direct limits of exact sequences are exact). The sets $FP_n(R\text{-Mod})$ have been intensely studied and in here $FP_n = FP_{n.5}$ for all n. The reason for introducing the classes $FP_{n.5}(\mathscr{A})$ is that they are more stable in that they are closed under extensions ([II, Lem. 1.3]), not just finite sums ([4, Lem. 1.3]).

The direct limit is very well behaved in locally finitely presented categories as the following lemma shows

Lemma 3. *Let $\mathscr{A}$ be a locally finitely presented category. Then*

1. *$\mathscr{A}$ is AB5 (Crawley-Boevey [6])*
2. *If $\mathscr{X} \subseteq FP_1(\mathscr{A})$ is closed under finite sums then $\varinjlim \mathscr{X}$ is closed under direct limits and summands (Lenzing [21]).*
3. *If $\mathscr{X} \subseteq FP_2(X)$ is closed under extensions so is $\varinjlim \mathscr{X}$ ([II, Prop. 1.1]).*

1.2 Multiplicative structure

Definition 4 ([23, XI])**.** A monoidal category is a triple $(\mathscr{A}, \otimes, 1)$ where $\mathscr{A}$ is a category, $\otimes\colon \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ is an associative bifunctor and $1 \in \mathscr{A}$ is a unit for $\otimes$ i.e. $1 \otimes X \cong X \cong X \otimes 1$, satisfying natural coherence diagrams. It is symmetric if we have a natural isomorphism $\lambda\colon X \otimes Y \cong Y \otimes X$ interacting with the associator and unit in a natural way.

When the category is monoidal we have another notion of smallness:

Definition 5. [Lewis, May and Steinberger [22, III.§1]] An object X in a symmetric monoidal category $\mathscr{A}$ is called *dualizable* if $X \otimes -$ has a right adjoint of the form $X^* \otimes -$ for some $X^* \in \mathscr{A}$.

When the category is closed, i.e $- \otimes X$ has a right adjoint $[X, -]$ for any $X \in \mathscr{A}$, then X always has a dual object $X^* = [X, 1]$. In this case X is dualizable precisely when $X^* \otimes - \cong [X, -]$ ([22, III.§1]). Again we are interested in categories built out of small objects. We say $\mathscr{A}$ is generated by dualizable objects if there is a set of dualizable objects generating $\mathscr{A}$, where a set $\mathcal{S}$ is said to generate $\mathscr{A}$ if a morphism $X \to Y$ is zero in $\mathscr{A}$ iff $S \to X \to Y$ is zero for every map $S \to X$ with $S \in \mathcal{S}$.

The additive and multiplicative structures may interact in the following way. First it is easy to see that if $\otimes$ is *continuous* (respects direct limits) and 1 is finitely presented (or even FP_n) then so is every dualizable object. Secondly as in [4, Satz 1.5] we see that if $\mathscr{A}$ is AB5 and is generated by a set of finitely presented objects, then it is locally finitely presented. So an AB5-abelian monoidal category with continuous tensor and finitely presented unit is locally finitely presented if it is generated by dualizable objects.

For dualizable objects in monoidal categories we have the following. For the last assertion see [I, Thm. 2]. The first is by the Yoneda lemma.

Proposition 6. *Let $\mathscr{A}$ be a monoidal category. An object $X \in \mathscr{A}$ is dualizable iff there is an object $X^* \in \mathscr{A}$ s.t*

$$\operatorname{Hom}(1, X^* \otimes -) \cong \operatorname{Hom}(X, -).$$

If X is dualizable so is X^* and $(-)^*\colon \mathscr{X} \to \mathscr{X}$ is a duality where $\mathscr{X}$ is the dualizable objects of $\mathscr{A}$. If $\mathscr{A}$ has enough $\otimes$-flat objects (e.g., it is generated by dualizable objects) then for any dualizable object $X \in \mathscr{A}$ we further have

$$\operatorname{Ext}(1, X^* \otimes -) \cong \operatorname{Ext}(X, -).$$

As mentioned in the introduction the theory of monoidal categories is not flexible enough for the non-commutative cases so we enlarge it as follows:

Definition 7 ([I, Setup 1])**.** A $\otimes$-abelian triple consits of three AB5-abelian categories ($\mathscr{A}$,$\mathscr{B}$,$\mathscr{C}$) together with a continuous bifunctor

$$\otimes\colon \mathscr{B} \times \mathscr{A} \to \mathscr{C}.$$

We call $\mathscr{A}$ (left) $\otimes$-abelian.

Inspired by Proposition 6 we define the dualizable objects of an $\otimes$-abelian triple as follows

Definition 8 (I, Setup 1+2)**.** Let $(\mathscr{A},\mathscr{B},\mathscr{C})$ be a $\otimes$-abelian triple and let $1 \in \mathrm{FP}_1(\mathscr{C})$. We say $X \in \mathscr{A}$ is *dualizable* if

$$\mathrm{Hom}(1, X^* \otimes -) = \mathrm{Hom}(X, -).$$

It is strongly dualizable if further

$$\mathrm{Ext}(1, X^* \otimes -) = \mathrm{Ext}(X, -).$$

Similarly for $Y \in \mathscr{B}$. We say $(\mathscr{A},\mathscr{B},\mathscr{C})$ is *generated by (strongly) dualizable* objects if there is a duality

$$(-)^*\colon \mathscr{X} \to \mathscr{X}^*$$

between full skeletally small[1] subcategories of (strongly) dualizable objects $\mathscr{X} \subseteq \mathscr{A}$ and (strongly) dualizable objects $\mathscr{X}^* \subseteq \mathscr{B}$ s.t. $\mathscr{X}$ generates $\mathscr{A}$ and $\mathscr{X}^*$ generates $\mathscr{B}$. In this case we say $\mathscr{A}$ is a left $\otimes$-abelian category with (strongly) dualizable generators $\mathscr{X}$.

As before the dualizable objects are finitely presented and if $\mathscr{A}$ is generated by dualizable objects it is locally finitely presented.

Example 9. If $(\mathscr{C}, \otimes, 1)$ is a (possibly non-symmetric) monoidal category, then $(\mathscr{C}, \mathscr{C}, \mathscr{C}, \otimes, 1)$ is a $\otimes$-triple. Further $(\mathscr{C}, \otimes, 1)$ is generated by dualizable objects in the sense of Definition 5 iff $(\mathscr{C}, \mathscr{C}, \mathscr{C}, \otimes, 1)$ is generated by dualizable objects in the sense of Definition 8. These are automatically strong.

The $\otimes-$abelian categories are stable under many operations used in abstract algebra. In particular, if $(\mathscr{A},\mathscr{B},\mathscr{C})$ is a $\otimes$-abelian triple, so is $(\mathrm{Ch}(\mathscr{A}), \mathrm{Ch}(\mathscr{B}), \mathrm{Ch}(\mathscr{C}))$ and $(\mathrm{Fun}(D, \mathscr{A}), \mathrm{Fun}(D^{\mathrm{op}}, \mathscr{B}), \mathscr{C})$ for any small category D. Another way of getting $\otimes$-abelian categories is to start from a symmetric monoidal category $\mathscr{C}$ with a ring-object A. Then $(A\text{-Mod}, A\text{-Mod}, \mathscr{C})$ will be a $\otimes$-abelian triple (see [I, Ex. 1 and Sec. 5] and [II, Ex. 4.2]). Using these procedures we get all the examples mentioned so far.

[1] the isomorphisms classes form a set

In many cases we will have more structure and similarly to Definition 5 have another tensor $\otimes_0 \colon \mathscr{A} \times \mathscr{C} \to \mathscr{A}$ s.t. $X \in \mathscr{A}$ is dualizable iff $X \otimes_0 -$ has a right adjoint given by $X^* \otimes_A -$ for some $X^* \in \mathscr{B}$. We will also have some global functor $(-)^* \colon \mathscr{A}^{\mathrm{op}} \to \mathscr{B}$, s.t. X is dualizable iff this particular X^* is the dual. This is for instance the case in A-Mod where A is a ring object in a symmetric monoidal closed category [I, Lem. 5]).

Examples of dualizable objects are proj(R-Mod) (finitely generated projective modules) in R-Mod, the perfect complexes in Ch(R-Mod), the finitely generated semi-projective objects in A-DGMod, the representable functors in Fun(D, R-Mod) where D is a small additive category and the locally free sheaves of finite rank in QCoh(X). These are all strong ([I, Sec. 5]).

1.3 Cotorsion pairs

In R-Mod the dualizable objects are projective, and the projective objects play a key role in homological algebra here. We can build projective resolutions that we, among many other things, can use to compute the Ext functor. What makes this work is that the class of projective objects is *precovering.*

Definition 10. A class $\mathcal{P}$ in a category $\mathscr{A}$ is said to be *precovering* if for every $M \in \mathscr{A}$ there is a map $\varphi \colon P \to M$, called the *precover* s.t. every map $Q \to M$ with $Q \in \mathcal{P}$ factors throug φ. I.e we can always complete the following diagram to a commutative one

$$\begin{array}{ccc} & & Q \\ & \swarrow & \downarrow \\ P & \xrightarrow{\varphi} & M \end{array} .$$

Whenever we have a precovering class $\mathcal{P}$, every object has a $\mathcal{P}$-resolution that we can use instead of the projective resolution to build a relative $\mathrm{Ext}_{\mathcal{P}}$ functor. In Ch(R-Mod) (and more generally A-DGMod) the dualizable objects are not projective and even though these categories have enough projective objects, the projective objects are homologically trivial so homological algebra using these is not necessarily very interesting. Here it is more fruitful to look at the so-called DG-projective objects (see Avramov, Foxby and Halperin [2] or [14] for the case of chain complexes). It might also happen, as in QCoh(X), that there simply are not enough projective objects.

To develop homological algebra in general $\otimes$-abelian categories we turn to the theory of cotorsion pairs.

Definition 11 (Salce [26])**.** Let $\mathscr{X}$ be a class of objects in an abelian category $\mathscr{C}$. We define

$$\mathscr{X}^{\perp} = \{Y \in \mathscr{C} \mid \forall X \in \mathscr{X} \colon \operatorname{Ext}^1_{\mathscr{C}}(X, Y) = 0\} \text{ and}$$
$${}^{\perp}\mathscr{X} = \{Y \in \mathscr{C} \mid \forall X \in \mathscr{X} \colon \operatorname{Ext}^1_{\mathscr{C}}(Y, X) = 0\}$$

We say $(\mathscr{A}, \mathscr{B})$ is a cotorsion pair if $\mathscr{A}^\perp = \mathscr{B}$ and ${}^\perp\mathscr{B} = \mathscr{A}$. It is *complete* if every $C \in \mathscr{C}$ has a presentation

$$0 \to B \to A \to C \to 0$$

with $A \in \mathscr{A}$ and $B \in \mathscr{B}$ and a presentation

$$0 \to C \to B' \to A' \to 0$$

with $A' \in \mathscr{A}$ and $B' \in \mathscr{B}$.

These conditions precisely ensures that the left part of a complete cotorsion pair is precovering so we can use it to build resolutions in the theory of relative homological algebra. Luckily there is a canonical way of getting a complete cotorsion pair:

Lemma 12 (Saorín and Šťovíček [27, Cor. 2.15 (2)])**.** *Let $\mathscr{X}$ be a generating set of objects in an AB5-abelian category $\mathscr{A}$. Then $({}^\perp(\mathscr{X}^\perp), \mathscr{X}^\perp)$ is a complete cotorsion pair. Further ${}^\perp(\mathscr{X}^\perp) = \mathrm{sFilt}\,\mathscr{X}$ where $\mathrm{sFilt}\,\mathscr{X}$ is the class of summands of $\mathscr{X}$-filtered objects (transfinite extensions of $\mathscr{X}$).*

This motivates the following definition.

Definition 13 ([I, Def. 6])**.** Let $\mathscr{A}$ be a $\otimes$-abelian category generated by dualizable objects $\mathscr{X}$. We define

$$(\mathscr{X}\text{-Proj}, \mathscr{X}\text{-acyclic}) = ({}^\perp(\mathscr{X}^\perp), \mathscr{X}^\perp).$$

When $\mathscr{X}$ is understood we call them the semi-projective and acyclic objects.

If $\mathscr{X}$ consists of projective objects, then this gives the *categorical cotorsion pair* $(\mathrm{Proj}\,\mathscr{A}, \mathscr{A})$. In $\mathrm{Ch}(R\text{-Mod})$ and A-DGMod we get the cotorsion pair of (DG-projective,acyclic) mentioned above. We also get an interesting one in $\mathrm{QCoh}(\mathscr{X})$, see ([I, Ex. 1. and Sec. 4]). In the first paper we study this cotorsion pair and generalizes a classic description of $\varinjlim \mathscr{X}$ by Lazard and Govorov.

Another important class in relative homological algebra that we will use is the class of Gorenstein projective objects. This was first defined in R-Mod (see Enochs and Jenda [8]) but generalizes straightforwardly to abelian categories (see for examples Holm [16]).

Definition 14. Let $\mathscr{A}$ be an abelian category. An object in $\mathscr{A}$ is called *Gorenstein projective* if it is of the form $\mathrm{Ker}(P_1 \to P_0)$ for some exact complex $P_\bullet \in \mathrm{Ch}(\mathrm{Proj}(\mathscr{A}))$ s.t. $\mathrm{Hom}(P_\bullet, P)$ is exact for all $P \in \mathrm{Proj}(\mathscr{A})$.

Even in R-mod it is not known in general whether the Gorenstein projective objects are the left part of a complete cotorsion pair or even precovering. We do however in many special cases. In the case of modules over a ring Holm

[16] proves that if an object has a finite Gorenstein projective resolution, then it has a Gorenstein projective precover.

Bravo, Gillespie and Hovey [3] proves that the Gorenstein projective objects are the left part of a complete cotorsion pair in many situations and suggests a modification of the above definition and show that the new so-called AC-Gorenstein projective objects ([3, Sec. 9]) are always the left part of a complete cotorsion pair. Their new definition agrees when R is right coherent and has finite finistic projective dimension (combine [16, Prop 2.3], [18, Prop. 6] and [3, Cor. 2.11]). Their definition generalizes readily to $\otimes$-abelian categories, but we will not need it here.

1.4 Pontryagin duals

In studying the various classes of projective objects the Pontryagin dual is a useful tool. In Ab it is defined as $(-)^+ = [-, \mathbb{Q}/\mathbb{Z}]\colon \mathrm{Ab} \to \mathrm{Ab}$, and there are similar ad hoc definitions in many other situations. They are all covered by the following definition:

Definition 15 ([II, Def. 4.1])**.** Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a $\otimes$-abelian triple. A Pontryagin dual consists of two functors

$$(-)^+ : \mathscr{A}^{\mathrm{op}} \to \mathscr{B}, \quad (-)^+ : \mathscr{B}^{\mathrm{op}} \to \mathscr{A}$$

that both create exactness (i.e $A \to B \to C$ is exact iff $C^+ \to B^+ \to A^+$ is exact) together with natural ismorphisms

$$\mathscr{A}(A, B^+) \cong \mathscr{C}(B \otimes A, E) \cong \mathscr{B}(B, A^+)$$

for some injective cogenerator $E \in \mathscr{C}$, i.e $\mathscr{C}(-, E)$ creates exactness.

As before if $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ has a Pontryagin dual in this sense, we get induced Pontryagin duals in $(\mathrm{Ch}(\mathscr{A}), \mathrm{Ch}(\mathscr{B}), \mathrm{Ch}(\mathscr{C}))$ and $(\mathrm{Fun}(D, \mathscr{A}), \mathrm{Fun}(D^{\mathrm{op}}, \mathscr{B}), \mathscr{C})$. If $(\mathscr{C}, \otimes, [-,-], 1)$ is a symmetric monoidal closed category then a Pontryagin dual of the $\otimes$-abelian triple $(\mathscr{C}, \mathscr{C}, \mathscr{C}, \otimes, 1)$ is of the form $[-, E]$ and induces a Pontryagin dual in $(A\text{-Mod}, A\text{-Mod}, \mathscr{C})$. The standard Pontryagin duals (also sometimes called the character modules) in R-Mod, $\mathrm{Ch}(R\text{-Mod})$, $\mathrm{Fun}(D, R\text{-Mod})$, and A-DGMod for any ring R and DGA A are all induced by the standard Pontryagin dual in Ab in this way ([II, Ex. 4.2]).

Dually to the projective objects we have the injective objects and immediately from the definition we see that $\mathrm{Proj}(\mathscr{A})^+ \subseteq \mathrm{Inj}(\mathscr{B})$. Defining $\mathrm{Flat}(\mathscr{A})$ as the objects in $\mathscr{A}$ s.t. $- \otimes X$ is exact, we see that

$$X^+ \in \mathrm{Inj}(\mathscr{B}) \iff X \in \mathrm{Flat}(\mathscr{A}).$$

We also have a dual notion of Gorenstein projective and a notion of Gorenstein flatness.

Definition 16. Let $\mathscr{B}$ be an abelian category. An object in $\mathscr{B}$ is called *Gorenstein injective* if it is of the form $\mathrm{Ker}(I_1 \to I_0)$ for some exact complex $I_\bullet \in \mathrm{Ch}(\mathrm{Inj}(\mathscr{B}))$ s.t. $\mathrm{Hom}(I, I_\bullet)$ is exact for all $I \in \mathrm{Inj}(\mathscr{B})$. Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a $\otimes$-abelian category. An object in $\mathscr{A}$ is called *Gorenstein flat* (GFlat) if it is of the form $\mathrm{Ker}(F_1 \to F_0)$ for some exact complex $F_\bullet \in \mathrm{Ch}(\mathrm{Flat}(\mathscr{A}))$ s.t. $I \otimes F_\bullet$ is exact for all $I \in \mathrm{Inj}(\mathscr{B})$.

However, since not all injective objects are necessarily the Pontryagin dual of a projective object, the Pontryagin dual of a Gorenstein projective object is not necessarily Gorenstein injective.

Also even though the dual is Gorenstein injective, the object might not be Gorenstin flat. We call an object $X \in \mathscr{A}$ *weakly Gorenstein flat* (wGFlat) if X^+ is Gorenstein injective, and we do have that $\mathrm{GFlat}(\mathscr{A}) \subseteq \mathrm{wGFlat}(\mathscr{A})$. In the second paper we study these classes in the category of quiver-representations.

1.5 Quiver representations

An important notion in R-Mod is that of a quiver representation (Gabriel [12]). This generalizes readily to ($\otimes$-)abelian categories (Holm and Jørgensen [17]). A quiver is a directed graph and a representation of a quiver Q in a category $\mathscr{A}$ is a diagram of shape Q in $\mathscr{A}$. The category of quiver representations $\mathrm{Rep}(Q, \mathscr{A})$ is equivalent to $\mathrm{Fun}(\bar{Q}, \mathscr{A})$ where $\bar{Q}$ is the path-category of Q, i.e. the category with objects the vertices of Q and with morphisms all paths og Q. As special instances of functor categories, if $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ is a $\otimes$-abelian triple so is $(\mathrm{Rep}(Q, \mathscr{A}), \mathrm{Rep}(Q^{\mathrm{op}}, \mathscr{B}), \mathscr{C})$.

The structure of $\mathrm{Rep}(Q, \mathscr{A})$ depends on the shape of Q and one such important notion is the following:

Definition 17 ([10])**.** A quiver Q is left-rooted if there exists no “infinite path” (i.e. infinite sequence of composable arrows) of the form

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet\,.$$

Dually we say Q is right-rooted when Q^{op} (same vertices but reversed arrows) is left-rooted.

In [1, II] Assem, Simson and Skowrński treat finite quivers with *admissible relations* and representations in R-mod. This generalizes readily to infinite quivers in abelian categories [III, App. A]. Let $\Bbbk$ be a commutative ring. A $\Bbbk$-linear relation is a formal $\Bbbk$-linear combination of paths with the same source and target. That is, it is a morphism in $\Bbbk\bar{Q}$, the free $\Bbbk$-linear category of $\bar{Q}$. A quiver Q with a set of relations R is denoted (Q, R). Any set of relations R generates a two-sided ideal (R) in $\Bbbk\bar{Q}$ and we may form the quotient category $\Bbbk\bar{Q}/(R)$. We call a relation admissible if it is in the arrow-ideal generated by all the arrows of Q.

A represention of a quiver Q with $\Bbbk$-linear relations R in a $\Bbbk$-linear category $\mathscr{A}$ (i.e. $\Bbbk = \mathbb{Z}$ and $\mathscr{A}$ is abelian) is a representation $F \in \operatorname{Rep}(Q, \mathscr{A}) \cong \operatorname{Fun}(\bar{Q}, \mathscr{A}) \cong \operatorname{Fun}_{\Bbbk}(\Bbbk\bar{Q}, \mathscr{A})$ s.t. $F(\rho) = 0$ for every $\rho \in R$. That is, the category of quiver representations $\operatorname{Rep}((Q,R), \mathscr{A})$ is equivalent to $\operatorname{Fun}(\Bbbk\bar{Q}/(R), \mathscr{A})$.

As before, if $\mathscr{A}$ is left $\otimes$-abelian so is $\operatorname{Fun}((Q,R), \mathscr{A})$, where the opposite quiver of (Q,R) is (Q^{op}, R). We may extend the definition of rooted in the following way:

Definition 18 ([III, Def. 4.1])**.** A quiver Q with relations R is right-rooted provided that for every infinite path of the form

$$\bullet \xrightarrow{a_1} \bullet \xrightarrow{a_2} \bullet \xrightarrow{a_3} \cdots,$$

we have $a_N \cdots a_1 \in (R)$ for some $N < \infty$. Again (Q,R) is left-rooted if (Q^{op}, R) is right-rooted.

We see that $\operatorname{Rep}((Q,\emptyset), \mathscr{A}) \cong \operatorname{Rep}(Q, \mathscr{A})$ and Q is left (resp. right) rooted in the sense of Definition 17 iff $(Q,\emptyset)$ is left (resp. right) rooted in the sense of Definition 18. Right-rooted quivers with relations are studied in the third paper where we calculate their atom spectrum.

1.6 Atom spectra

Another very useful tool in R-Mod when R is commutative is the notion of a prime ideal. Among many other things they give the following correspondence in terms of the space of prime ideals $\operatorname{Spec} R$.

Theorem 19. *Let R be a commutative noetherian ring.*

1. *There is a bijective correspondence between* $\operatorname{Spec} R$ *and indecomposable injective objects of* R-Mod. *(Matlis [24])*

2. *There is a bijective correspondance between specialization closed subset of* $\operatorname{Spec} R$, *localizing subcategories of* R-Mod *and Serre subcategories of* R-mod *(Gabriel [13]).*

When R is non-commutative the many descriptions of prime ideals that are equivalent in the commutative case splits into different classes that all fail to give the above descriptions. Another approach is that of an atom. The idea goes back to Storrer [29]). The definition is inspired by the following observation

Observation 20 ([19, Prop 7.1]). Let R be a commutative ring and, let $\mathfrak{p}$ be a prime ideal and let N be a submodule of $R/\mathfrak{p}$. If N is non-trivial, then $R/\mathfrak{p}$ and $(R/\mathfrak{p})/N$ have no common submodule. Indeed let M be a common submodule of R/p and $(R/\mathfrak{p})/N$ and assume there is a non-zero element $a \in M$. Then $\operatorname{Ann}(a) = \mathfrak{p}$ as M is a submodule of $R/\mathfrak{p}$ and $\mathfrak{p}$ is prime. On the other hand

$N \cong I/\mathfrak{p}$ for some ideal $I \supseteq \mathfrak{p}$ and M is a submodule of $(R/\mathfrak{p})/N \cong R/I$ so $I \subseteq \mathrm{Ann}(a) \subseteq \mathfrak{p}$, hence N is trivial.

This leads us to the following definition:

Definition 21 ([19])**.** Let $\mathscr{A}$ be an abelian category. An object $H \in \mathscr{A}$ is *monoform* if for every non-zero $N \subseteq H$ there is no common subobject of H and H/N. Two monoform objects are equivalent if they have a common subobject. The equivalence classes of monoform objects form a topological space, $\mathrm{ASpec}\,\mathscr{A}$, called the atom spectrum of $\mathscr{A}$.

We call an ideal $\mathfrak{p}$ of a (possibly non-commutative ring) R comonoform if $R/\mathfrak{p}$ is monoforn. Then every atom is induced by a monoform object ([19, Prop. 6.2][2]), and if R is commutative there is a homeomorphism $\mathrm{Spec}\,R \cong \mathrm{ASpec}\,R\text{-Mod}$, when $\mathrm{Spec}\,R$ is equiped with the *Hochster dual* of the Zariski topology [19, Prop. 7.2][2].

Using the atom spectrum we can extend the correspondence for commutative noetherian rings to locally noetherian Grothendieck categories (including R-Mod when R is a right noetherian ring) as follows:

Theorem 22 ([19, Thm. 5.5 and 5.9])**.** *Let $\mathscr{A}$ be a locally noetherian Grothendieck category.*

1. *There is a bijective correspondence between* $\mathrm{ASpec}\,\mathscr{A}$ *and indecomposable injective objects of $\mathscr{A}$.*
2. *There is a bijective correspondance between open subsets of* $\mathrm{ASpec}\,\mathscr{A}$*, localizing subcategories of $\mathscr{A}$ and Serre subcategories of* $\mathrm{noeth}\,\mathscr{A}$*.*

In order for the atom spectrum to be truely useful we must know whether we can calculate it and if the description in these terms provides new insight. There have so far not been many concrete calculations in the literature and this is the topic of the last paper.

2 Introduction to paper I

Given a (left) $\otimes$-abelian category $\mathscr{A}$ with dualizable generators $\mathscr{X}$, (Definition 8) we have $\mathrm{sFilt}\,\mathscr{X} = \mathscr{X}\text{-Proj}$ since $\mathscr{X}\text{-Proj}$ is the left part of the cotorsion pair generated by $\mathscr{X}$ and we have $\mathrm{colim}\,\mathscr{X} = \mathscr{A}$ since $\mathscr{A}$ is locally finitely presented and every finitely presented $F \in \mathscr{A}$ has a presentation $X_1 \to X_0 \to F$ [4]. But what about $\varinjlim \mathscr{X}$?

Lazard [20] and independently Govorov [15] have shown that $\varinjlim \mathscr{X} = \mathrm{Flat}(\mathscr{X})$ when $\mathscr{A} = R\text{-Mod}$. In [I] we show that this always happens when the dualizable generators are projective [I, Cor. 1]. If they are not, we look to DGAs for inspiration ([2]) and define

[2]It is a standing assumption in [19, Sec. 6 and 7] that the rings are noetherian, but it is not needed for these propositions .

Definition 23 ([I, Def. 6]). Let $\mathscr{A}$ be a (left) $\otimes$-abelian category with dualizable generators $\mathscr{X}$. We say F is $\mathscr{X}$-flat if it is flat and

$$\mathscr{X}^{\perp} \otimes F \subseteq 1^{\perp}.$$

When $\mathscr{X}$ is understood we call such objects semi-flat.

The main result of [I] is then

Theorem 24 ([I, Thm. 1]). *Let $\mathscr{A}$ be a (left) $\otimes$-abelian category with strongly dualizable generators $\mathscr{X}$ s.t. the unit is FP_2. Then $\varinjlim \mathscr{X}$ is precisely the $\mathscr{X}$-flat objects.*

Even if the unit is not FP_2 and the dualizable generators are not strong every $\mathscr{X}$-flat object is still in the direct limit closure. If the generators are not strongly dualizable they might not be semi-flat, if the unit is not FP_2 the semi-flat objects might not be closed under direct limits.

Theorem 24 not only reproves the original theorem of Lazard and Govorov, but also the version for functor-categories by Oberst and Röhl [25], the version in Ch(R-Mod) by Christensen and Holm [5] and give the following new results:

Corollary 25. *Let R be a graded ring, and let $\mathscr{X}$ be the finitely generated projective graded modules. Then $\varinjlim \mathscr{X}$ is precisely the flat graded modules.*

Corollary 26. *Let $\mathscr{A}$ be a differential graded algebra and let $\mathscr{X}$ be the finitely generated semi-projective DG-modules, i.e. the summands of finite extensions of shifts of A. The cotorsion pair (DG-Proj,acyclic) is generated by $\mathscr{X}$ and $\varinjlim \mathscr{X}$ is precisely the semi-flat (i.e. DG-flat) objets.*

Corollary 27. *Let X be a noetherian scheme with the strong resolution property and let $\mathscr{X}$ be the locally free sheaves of finite rank. Then $\varinjlim \mathscr{X}$ is precisely the semi-flat sheaves.*

Before proving the result we develop the theory of $\otimes$-abelian categories in detail and show how all the mentioned examples fit in, with a focus on categories of modules over ring objects in symmetric monoidal categories. We do not use the term $\otimes$-abelian in [I], nor strongly dualizable, but refer to [I, Setup 1 and Setup 2].

3 Introduction to paper II

In this paper we describe various classes of objects in Rep($Q, \mathscr{A}$). In the following let ($\mathscr{A}, \mathscr{B}, \mathscr{C}$) be a $\otimes$-abelian triple with a Pontryagin dual, Q a left-rooted quiver and let $\mathscr{A}$ be locally finitely presented. Less can be assumed in some cases; see the precise statements in [II].

In R-Mod we have the following descriptions. See Enochs, Oyonarte and Torrecillas [10], Enochs and Estrada [7] and Eshraghi, Hafezi and Salarian [11].

Proposition 28. *Let $\mathscr{A} = R$-Mod for some ring R. Then*

$$\mathrm{Proj}(\mathrm{Rep}(Q,\mathscr{A})) = \Phi(\mathrm{Proj}(\mathscr{A})) \tag{1}$$
$$\mathrm{Flat}(\mathrm{Rep}(Q,\mathscr{A})) = \Phi(\mathrm{Flat}(\mathscr{A})) \tag{2}$$
$$\mathrm{GProj}(\mathrm{Rep}(Q,\mathscr{A})) = \Phi(\mathrm{GProj}(\mathscr{A})). \tag{3}$$

When R is Iwanaga-Gorenstein[3]

$$\mathrm{GFlat}(\mathrm{Rep}(Q,\mathscr{A})) = \Phi(\mathrm{GFlat}(\mathscr{A})) \tag{4}$$

where for any $\mathscr{X} \subseteq \mathscr{A}$ we define

$$\Phi(\mathscr{X}) = \left\{ F \in \mathrm{Rep}(Q,\mathscr{A}) \;\middle|\; \begin{array}{c} \forall v \colon \bigoplus_{w\to v} F(w) \to F(v) \\ \textit{is monic and has cokernel in } \mathscr{X} \end{array} \right\}.$$

(1) has been generalized to abelian categories with enough projective objects using cotorsion pairs (see [17]), and the proof of (3) works in any abelian category. The original proof of (1) and the proof of (2) construct certain filtration for elements of $\Phi(\mathrm{Add}\,\mathscr{X})$ and $\Phi(\varinjlim \mathscr{X})$ where $\mathscr{X} = \mathrm{proj}\,\mathscr{A}$ are the finitely generated projective objects of $\mathscr{A} = R$-Mod. A general description of $\Phi(\mathscr{X})$ seems in order and is given in the second paper as follows:

Theorem 29 ([II, Thm. A])**.**
Let $\mathscr{X} \subseteq \mathscr{A}$ be be arbitrary. Then

$$\begin{aligned} \Phi(\mathscr{X}) &\subseteq \mathrm{Filt}\, f_*(\mathscr{X}), \\ \Phi(\mathrm{Filt}\,\mathscr{X}) &= \mathrm{Filt}\, f_*(\mathscr{X}) = \mathrm{Filt}\,\Phi(\mathscr{X}) \textit{ and} \\ \Phi(\mathrm{sFilt}\,\mathscr{X}) &= \mathrm{sFilt}\, f_*(\mathscr{X}) = \mathrm{sFilt}\,\Phi(\mathscr{X})^4. \end{aligned}$$

If $\mathscr{X} \subseteq FP_{2.5}(\mathscr{A})$ is closed under extensions, then

$$\Phi(\varinjlim \mathscr{X}) = \varinjlim \mathrm{ext}\, f_*(\mathscr{X}) = \varinjlim \Phi(\mathscr{X})$$

where

$$f_*(\mathscr{X}) = \{f_v(X) \mid v \in Q, X \in \mathscr{X}\}$$

and $f_v \colon \mathscr{A} \to \mathrm{Rep}(Q,\mathscr{A})$ is the left adjoint to evaluation at the vertex v.

Using this we get the description (1) from Proposition 28 when $\mathscr{A}$ enough projective objects and (2) in the form $\varinjlim \mathrm{proj}\,\mathrm{Rep}(Q,\mathscr{A}) = \Phi(\varinjlim \mathrm{proj}(\mathscr{A}))$ when $\mathrm{proj}\,\mathscr{A}$ generate $\mathscr{A}$. Using the Pontryagin dual we can characterize the

[3]see Enochs and Jenda [9, Def. 9.11]
[4]If $\mathscr{X}$ is a generating set this can also be achieved using cotorsion pairs as in [17]

flat (and weakly flat Gorenstein) quiver representations in general (see below). The characterization of the Gorenstein flat representations uses that over Gorenstein rings the Gorenstein flat and weakly Gorenstein flat representations coincide. In general these classes are different, and different from the direct limit closure of $\mathrm{Gproj}(\mathscr{A}) := \mathrm{GProj}(\mathscr{A}) \cap FP_{2.5}(\mathscr{A})$. Their characterization depend on the shape of Q and we define:

Definition 30 ([II, Def 2.1])**.** A quiver Q is *locally path-finite* if there are only finitely many paths between any two given verticies. It is *target-finite* if there are only finitely many arrows with a given target.

We now have

Theorem 31 ([II, Thm. B+C and Prop. 5.6])**.**

$$\mathrm{Flat}(\mathrm{Rep}(Q,\mathscr{A})) = \Phi(\mathrm{Flat}(\mathscr{A}))$$
$$\mathrm{wGFlat}(\mathrm{Rep}(Q,\mathscr{A})) = \Phi(\mathrm{wGFlat}(\mathscr{A}))$$

If Q is target-finite and locally path-finite and $\mathscr{A}$ has enough projective objects, then

$$\varinjlim \mathrm{Gproj}(\mathrm{Rep}(Q,\mathscr{A})) = \Phi(\varinjlim \mathrm{Gproj}(\mathscr{A})).$$

If Q is target-finite and

1. *Products in $\mathscr{A}$ preserve epis and flatness,*
2. $\mathrm{Inj}(\mathscr{B})^+ \subseteq \mathrm{Flat}(\mathscr{A})$,
3. $\mathrm{proj}(\mathscr{A})$ *generate* $\mathscr{A}$,

then

$$\mathrm{wGFlat}(\mathrm{Rep}(Q,\mathscr{A})) = \mathrm{GFlat}(\mathrm{Rep}(Q,\mathscr{A})).$$

If $\mathscr{A} = R$-Mod the last three conditions precisely says that R is right coherent.

In particular we extend (5) to right coherent rings and show that under the conditions of Theorem 31, the condition $\varinjlim \mathrm{Gproj} = \mathrm{GFlat}$ lifts from $\mathscr{A}$ to $\mathrm{Rep}(Q,\mathscr{A})$. Similarly with the condition $\varinjlim \mathrm{Gproj} = \mathrm{wGFlat}$.

This paper explains the basic facts on quivers (without relations) and introduce the classes of $FP_{n.5}$ objects and the abstract of Pontryagin dual. The paper uses definitions for Pontryagin duals and Gorenstein flat objects that work in any abelian, not necessarily $\otimes$-abelian category. The statements are proven in this generality and specializes to those given in this introduction.

4 Introduction to paper III

In this paper we show how to calculate the atom spectrum of a category using what we here call tilings. A functor $F\colon \mathscr{A} \to \mathscr{B}$ is said to *lift subobjects* ([III, Def. 3.1]) if for every $X \in A$, every subobject of $F(X)$ is the image of a subobject of X.

Definition 32. Let $\mathscr{B}$ be a category. A collection of fully faithful and exact functors $\{F_i\colon \mathscr{A}_i \to \mathscr{B}\}$ that lifts subobjects is called a *tiling* if the following three conditions hold:

1. Each F_i has a right adjoint G_i s.t. $F_iG_iB \to B$ is monic for every $B \in \mathscr{B}$.
2. If $G_iB = 0$ for every i, then $B = 0$
3. If F_iA_i and F_jA_j have a common non-zero subobject then $i = j$.

The conditions say that the tiles cannot be deformed, they cover the whole category and they cannot overlap. When a category has a tiling every atom lies in one of the tiles

Theorem 33 ([III, Thm. 3.7])**.** *Let $\mathscr{B}$ be an abelian category and let $\{F_i\colon \mathscr{A}_i \to \mathscr{B}\}$ be a tiling of abelian categories. Then*

$$\operatorname{ASpec}\mathscr{B} \cong \coprod_i \operatorname{ASpec}\mathscr{A}_i$$

both as a set, ordered set and topological space. The map sends the equivalence class of the monoform $A_i \in \mathscr{A}_i$ to the equivalence class of $F_i(A_i)$.

We give two abstract examples of tilings:

Proposition 34 ([III, Thm. 4.9])**.** *Let $\mathscr{A}$ be a $\Bbbk$-linear abelian category and (Q, R) a right-rooted quiver with $\Bbbk$-linear admissible relations.*

Then $\{S_i\colon \mathscr{A} \to \operatorname{Rep}((Q,R),\mathscr{A})\}_{i\in Q}$ is a tiling, where $S_i(A)$ is the stalk representation $S_i(A)(i) = A$ and $S_i(A)(j) = 0$ when $i \neq j$.

Example 35. To see why right-rooted and admissible matters in Proposition 34 we will look at a simple commutative example with a commutative ring $\Bbbk$ and the Jordan quiver

$$Q: \quad \bullet \circlearrowleft$$

In this case $\operatorname{Rep}(Q, \Bbbk\text{-Mod}) \cong \Bbbk[x]\text{-Mod}$, so

$$\operatorname{ASpec}(\operatorname{Rep}(Q, \Bbbk\text{-Mod})) \cong \operatorname{ASpec}(\Bbbk[x]\text{-Mod}) \cong \operatorname{Spec}\Bbbk[x].$$

The ring $\Bbbk[x]$ has many interesting primes not coming from primes of $\Bbbk$ which is reflected by the fact that Q is not right-rooted. To make it right-rooted we have to impose a relation of the form x^n. This precisely kills all interesting primes, i.e.

$$\operatorname{ASpec}(\operatorname{Rep}((Q,\{x^n\}),\Bbbk)) \cong \operatorname{Spec}(\Bbbk[x]/(x^n)) \cong \operatorname{Spec}\Bbbk.$$

That nothing else is killed is reflected by the fact that the imposed relation is admissible, i.e. contained in the arrow-ideal which in this case is just (x). A non-admissible relation corresponds to a polynomial, f, s.t. $f(0) \neq 0$ and we see that

$$\operatorname{ASpec}(\operatorname{Rep}((Q,\{x^n,f\}),\Bbbk)) \cong \operatorname{Spec}(k[x]/(x^n,f)) \cong \operatorname{Spec}(\Bbbk/f(0)).$$

That is, any non-admissible relation kills some primes of $\Bbbk$. To have an equality as in Theorem 33 thus precisely requires (Q, R) to be right-rooted with admissible relations. The theorem says that this is a general phenomena.

Another tiling is of comma categories ([23, II.6]). Let

$$\mathscr{A} \xrightarrow{U} \mathscr{C} \xleftarrow{V} \mathscr{B}$$

be a diagram of categories. The comma category $(U \downarrow V)$ is the category of triples $(A \in \mathscr{A}, B \in \mathscr{B}, \theta \colon UA \to VB)$ with morphisms
$(\alpha \colon A \to A', \beta \colon B \to B')$ s.t. the following diagram commutes:

$$\begin{array}{ccc} UA & \xrightarrow{U\alpha} & UA' \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta'} \\ VB & \xrightarrow{V\beta} & VB' \end{array}.$$

The comma category of abelian categories is not always abelian, but when it is we have the following tiling.

Proposition 36 ([III, Prop. 5.1 + Thm. B])**.** *Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be abelian categories and assume $U \colon \mathscr{A} \to \mathscr{C}$ has a right adjoint and $V \colon \mathscr{B} \to \mathscr{C}$ is left exact. Then $(U \downarrow V)$ is abelian and $(A \mapsto (A, 0, 0), B \mapsto (0, B, 0))$ is a tiling.*

As an example of using comma categories [III, Ex. 5.4], the comonoform ideals of the generalized matrix ring

$$T = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix},$$

where A and B are commutative rings and M is a (B, A)-bimodule, are all of the form

$$\begin{pmatrix} \mathfrak{p} & 0 \\ M & B \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A & 0 \\ M & \mathfrak{q} \end{pmatrix}$$

for primes $\mathfrak{p}$ of A and $\mathfrak{q}$ of B, and all the induced atoms are different.

In this paper we introduce the reader to the theory of atoms and of quivers with relations. We then give concrete examples of computations of the atom spectra of the module category of non-commutative rings.

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Paper I

Dualizable and semi-flat objects in abstract module categories

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DUALIZABLE AND SEMI-FLAT OBJECTS IN ABSTRACT MODULE CATEGORIES

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Abstract. In this paper, we define what it means for an object in an abstract module category to be dualizable and we give a homological description of the direct limit closure of the dualizable objects. Our description recovers existing results of Govorov and Lazard, Oberst and Röhrl, and Christensen and Holm. When applied to differential graded modules over a differential graded algebra, our description yields that a DG-module is semi-flat if and only if it can be obtained as a direct limit of finitely generated semi-free DG-modules. We obtain similar results for graded modules over graded rings and for quasi-coherent sheaves over nice schemes.

1. Introduction

In the literature, one can find several results that describe how some kind of "flat object" in a suitable category can be obtained as a direct limit of simpler objects. Some examples are:

(1) In 1968 Lazard [22], and independently Govorov [11] proved that over any ring, a module is flat if and only if it is a direct limit of finitely generated projective modules.
(2) In 1970 Oberst and Röhrl [25, Thm 3.2] proved that an additive functor on a small additive category is flat if and only if it is a direct limit of representable functors.
(3) In 2014 Christensen and Holm [5] proved that over any ring, a complex of modules is semi-flat if and only if it is a direct limit of perfect complexes (= bounded complexes of finitely generated projective modules).
(4) In 1994 Crawley-Boevey [6] proved that over certain schemes, a quasi-coherent sheaf is locally flat if and only if it is a direct limit of locally free sheaves of finite rank. In 2014 Brandenburg [3] defined another notion of flatness and proved one direction for more general schemes.

In Section 3 we provide a categorical framework that makes it possible to study results and questions like the ones mentioned above. It is this framework that the term "abstract module categories" in the title refers to. From a suitably nice (axiomatically described) class $\mathcal{S}$ of objects in such an abstract module category $\mathscr{C}$, we define a notion of semi-flatness (with respect to $\mathcal{S}$). This definition depends only on an abstract tensor product, which is built into the aforementioned framework, and on a certain homological condition. We write $\varinjlim \mathcal{S}$ for the class of objects in $\mathscr{C}$ that can be obtained as a direct limit of objects from $\mathcal{S}$. Our main result shows that under suitable assumptions, $\varinjlim \mathcal{S}$ is precisely the class of semi-flat objects:

Theorem 1. *Let $\mathscr{C}$ and $\mathcal{S}$ be as in Setup 1 and Setup 2. In this case, an object in $\mathscr{C}$ is semi-flat if and only if it belongs to $\varinjlim \mathcal{S}$.*

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The proof of this theorem is a generalization of the proof of [5, Thm. 1.1], which in turn is modelled on the proof of [22, Chap. I, Thm. 1.2]. A central new ingredient in the proof of Theorem 1 is an application of the generalized Hill Lemma by Stovicek [32, Thm 2.1].

The abstract module categories treated in Section 3 encompass more "concrete" module categories such as the category ${}_A\mathscr{C}/\mathscr{C}_A$ of left/right modules over a monoid (= ring object) A in a closed symmetric monoidal abelian category $(\mathscr{C}_0, \otimes_1, 1, [-,-])$; see Pareigis [26]. In this setting, Theorem 1 takes the form:

Theorem 2. *Let A be a monoid in a closed symmetric monoidal Grothendieck category $(\mathscr{C}_0, \otimes_1, [-,-], 1)$ and let ${}_A\mathscr{C}/\mathscr{C}_A$ be the category of left/right A-modules. Let ${}_A\mathcal{S}$ be (a suitable subset of, e.g. all) the* dualizable *objects in ${}_A\mathscr{C}$. If $\mathscr{C}_0$ is generated by dualizable objects and 1 is* FP_2*, then the direct limit closure of ${}_A\mathcal{S}$ is precisely the class of semi-flat objects in ${}_A\mathscr{C}$.*

Dualizable objects in symmetric monoidal categories were defined and studied by Lewis and May in [23, III§1] and investigated further by Hovey, Palmieri, and Strickland in [19]; we extend the definition and the theory of such objects to categories of A-modules (see Definition 7).

In the final Section 5, we specialize our setup even further. For some choices of a closed symmetric monoidal abelian category $\mathscr{C}_0$ and of a monoid $A \in \mathscr{C}_0$, the category of A-modules turn out to be a well-known category in which the dualizable and the semi-flat objects admit hands-on descriptions. When applied to differential graded modules over a DGA, to graded modules over a graded ring, and to sheaves over a scheme, Theorem 2 yields the following results, which all seem to be new.

Theorem 3. *Let $\mathcal{S}$ be the class of finitely generated semi-free/semi-projective differential graded modules over a differential graded algebra A. The direct limit closure of $\mathcal{S}$ is precisely the class of semi-flat (or DG-flat) differential graded A-modules.*

Corollary 4. *Over any $\mathbb{Z}$-graded ring, the direct limit closure of the finitely generated projective (or free) graded modules is precisely the class of flat graded modules.*

Theorem 4. *Let X be a noetherian scheme with the strong resolution property. In the category* $\mathrm{QCoh}(X)$*, the direct limit closure of the locally free sheaves of finite rank is precisely the class of semi-flat sheaves.*

In the same vein, it follows that the results (1)–(3), mentioned in the beginning of the Introduction, are also consequences of Theorems 1 and 2.

2. Preliminaries

2.1. **Locally finitely presented categories.** We need some facts about locally finitely presented categories from Breitsprecher [4]. Let $\mathscr{C}$ be a category. First recall:

Definition 1. A collection of objects $\mathcal{S}$ is said to *generate* $\mathscr{C}$ if given different maps $f, g\colon A \to B$ there exists a map $\sigma\colon S \to A$ with $S \in \mathcal{S}$ such that $f\sigma$ and $g\sigma$ are different. If $\mathscr{C}$ is abelian, this simply means that if $A \to B$ is non-zero there is some $S \to A$ with $S \in \mathcal{S}$ such that $S \to A \to B$ is non-zero.

Definition 2. An object $K \in \mathscr{C}$ is called *finitely presented* if $\mathscr{C}(K, -)$ commutes with filtered colimits. Denote by $\mathrm{fp}(\mathscr{C})$ the collection of all finitely presented objects in $\mathscr{C}$. A Grothendieck category is called *locally finitely presented* if it is generated by a small set (as opposed to a class) of finitely presented objects.

Remark 1. By [4, SATZ 1.5] a Grothendieck category is locally finitely presented if and only if $\varinjlim \mathrm{fp}(\mathscr{C}) = \mathscr{C}$, and by [6, (2.4)] this is equivalent to saying that $\mathscr{C}$ is abelian, $\mathrm{fp}(\mathscr{C})$ is small, and $\varinjlim \mathrm{fp}(\mathscr{C}) = \mathscr{C}$.

Proposition 1. *Let $\mathscr{C}$ be a Grothendieck category. Then*

1. [4, SATZ 1.11] *If $\mathcal{S}$ is a set of finitely presented objects generating $\mathscr{C}$, then $N \in \mathscr{C}$ is finitely presented iff it has a presentation*
$$X_0 \longrightarrow X_1 \longrightarrow N \longrightarrow 0$$
where X_0, X_1 are finite sums of elements of $\mathcal{S}$.
2. [4, SATZ 1.9] *The finitely presented objects are closed under extensions.*

Next we look at some properties of the class $\varinjlim \mathcal{S}$ of objects in $\mathscr{C}$ that can be obtained as a direct limit of objects from $\mathcal{S}$.

Lemma 1. [6, Lemma p. 1664] *Let $\mathscr{C}$ be a locally finitely presented Grothendieck category, let $M \in \mathscr{C}$ and let $\mathcal{S}$ be a collection of finitely presented objects closed under direct sums. If any map from a finitely presented object to M factors through some $S \in \mathcal{S}$, then $M \in \varinjlim \mathcal{S}$. In particular $\varinjlim \mathcal{S}$ is closed under direct limits and direct summands.*

Remark 2. Notice that the converse is true by definition for any $\mathcal{S}$ and $\mathscr{C}$.

We will later need the following way of extending the defining isomorphism of an adjunction to the level of Exts.

Lemma 2. [16, Lem. 5.1] *Let $F\colon \mathscr{C} \leftrightarrows \mathscr{D} : G$ be an adjunction of abelian categories, where F is left adjoint of G, and let $A \in \mathcal{C}$ be an object. If G is exact and if F leaves every short exact sequence $0 \to A' \to E \to A \to 0$ (ending in A) exact, then there is a natural isomorphism* $\mathrm{Ext}^1_{\mathscr{D}}(FA, -) \cong \mathrm{Ext}^1_{\mathscr{C}}(A, G-)$.

2.2. **Cotorsion pairs.** The theory of cotorsion pairs goes back to Salce [27] and has been intensively studied. See for instance Göbel and Trlifaj [10].

Definition 3. Let $\mathcal{X}$ be a class of objects in an abelian category $\mathscr{C}$. We define

- $\mathcal{X}^{\perp} = \{Y \in \mathscr{C} \mid \forall X \in \mathcal{X}\colon \mathrm{Ext}^1_{\mathscr{C}}(X, Y) = 0\}$
- ${}^{\perp}\mathcal{X} = \{Y \in \mathscr{C} \mid \forall X \in \mathcal{X}\colon \mathrm{Ext}^1_{\mathscr{C}}(Y, X) = 0\}$

Definition 4. Let $\mathcal{A}$ and $\mathcal{B}$ be classes of objects in an abelian category $\mathscr{C}$. We say $(\mathcal{A}, \mathcal{B})$ is a *cotorsion pair*, if $\mathcal{A}^{\perp} = \mathcal{B}$ and ${}^{\perp}\mathcal{B} = \mathcal{A}$. It is *complete* if every $C \in \mathscr{C}$ has a presentation
$$0 \to B \to A \to C \to 0$$
with $A \in \mathcal{A}$ and $B \in \mathcal{B}$ and a presentation
$$0 \to C \to B' \to A' \to 0$$
with $A' \in \mathcal{A}$ and $B' \in \mathcal{B}$. In this paper, we are only concerned with the first presentation.

Definition 5. An $\mathcal{S}$-filtration of an object X in a category $\mathscr{C}$ for a class of objects $\mathcal{S}$ is a chain
$$0 = X_0 \subseteq \cdots \subseteq X_i \subseteq \cdots \subseteq X_\alpha = X$$
of objects in $\mathscr{C}$ such that every X_{i+1}/X_i is in $\mathcal{S}$, and for every limit ordinal $\alpha' \leq \alpha$ one has $\varinjlim_{i<\alpha'} X_i = X_{\alpha'}$. An object X called $\mathcal{S}$-filtered if it has an $\mathcal{S}$-filtration. If $\alpha = \omega$ we say the filtration is countable, and if $\alpha < \omega$ that it is finite. In the latter case we will also say that X is a finite extension of $\mathcal{S}$.

Proposition 2. *If $\mathcal{S}$ is any generating set of objects in a Grothendieck category, then $({}^{\perp}(\mathcal{S}^{\perp}), \mathcal{S}^{\perp})$ is a complete cotorsion pair, and the objects in ${}^{\perp}(\mathcal{S}^{\perp})$ are precisely the direct summands of $\mathcal{S}$-filtered objects.*

Proof. See Saorín and Šťovíček [28, Exa. 2.8 and Cor. 2.15] (for the last assertion also see Šťovíček [32, Prop. 1.7]). ☐

When $\mathscr{C}$ is locally finitely presented and $\mathcal{S}$ consists of finitely presented objects and is closed under extensions, we can in fact realize any $S \in {}^{\perp}(\mathcal{S}^{\perp})$ as a direct limit. This generalizes the idea that an arbitrary direct sum can be realized as a direct limit of finite sums. The tool that allows us to generalize this idea is the generalized Hill Lemma. The full statement is rather technical so we just state here what we need (hence "weak version"):

Lemma 3 (Hill Lemma – weak version)**.** [32, Thm 2.1] *Let $\mathscr{C}$ be a locally finitely presented Grothendieck category, $\mathcal{S}$ be a set of finitely presented objects, and assume X has an $\mathcal{S}$-filtration. Given any map $f\colon S \to X$ from a finitely presented object, then $\operatorname{Im}(f) \subseteq S' \subseteq X$ for some finite extension S' of elements of $\mathcal{S}$.*

We can now prove:

Proposition 3. *Let $\mathcal{S}$ be a skeletally small class of finitely presented objects closed under finite extensions in a locally finitely presented Grothendieck category $\mathscr{C}$. Then any $\mathcal{S}$-filtered object is a direct limit of objects from S. In particular, ${}^{\perp}(\mathcal{S}^{\perp}) \subseteq \varinjlim \mathcal{S}$ when $\mathcal{S}$ generates $\mathscr{C}$.*

Proof. Let X be an $\mathcal{S}$-filtered object. Since $\mathscr{C}$ is locally finitely presented, X is also the direct limit of finitely presented objects X_i, hence also the direct limit of its finitely generated subobjects (images of finitely presented objects), but these are majored by $\mathcal{S}$-subobjects by Lemma 3, since $\mathcal{S}$ is closed under finite extensions. The last statement follows from Proposition 2 and Lemma 1. □

3. Abstract module categories

The aim in this section is to desribe the direct limit closure of $\mathcal{S}$ in the following setup:

Setup 1. Let $\mathscr{C}_L$, $\mathscr{C}_0$ and $\mathscr{C}_R$ be Grothendieck categories, let $\mathcal{S}_L \subseteq \mathscr{C}_L$ and $\mathcal{S}_R \subseteq \mathscr{C}_R$ be generating sets closed under extensions, and let $1 \in \mathscr{C}_0$ be finitely presented. Assume that we have a right continuous bifunctor (i.e. it preserves direct limits in each variable)

$$- \otimes -\colon \mathscr{C}_R \times \mathscr{C}_L \to \mathscr{C}_0$$

and a natural duality

$$(-)^*\colon \mathcal{S}_L \to \mathcal{S}_R$$

such that for any $S \in \mathcal{S}_L$ we have natural isomorphisms (also natural in S):

$$\mathscr{C}_0(1, S^* \otimes -) \cong \mathscr{C}_L(S, -) \quad \text{and}$$
$$\mathscr{C}_0(1, - \otimes S) \cong \mathscr{C}_R(S^*, -)$$

which is then analogously true for any $S \in \mathcal{S}_R$ by the duality between $\mathcal{S}_L$ and $\mathcal{S}_R$. For simplicity we will often write $\mathscr{C}$ for either $\mathscr{C}_L$ or $\mathscr{C}_R$ and $\mathcal{S}$ for either $\mathcal{S}_L$ and $\mathcal{S}_R$ (see for example Theorem 1). Hopefully this should not cause any confusion.

Remark 3. Note that in Setup 1 any $S \in \mathcal{S}$ is finitely presented because 1 is finitely presented and $\otimes$ is right continuous, so $\mathscr{C}_L$ and $\mathscr{C}_R$ are necessarily locally finitely presented. When there are notational differences we will work with $\mathscr{C}_L$ though everything could be done for $\mathscr{C}_R$ instead.

Example 1. Some specific examples of Setup 1 to have in mind are:

(1) A is a ring, $\mathscr{C}_L/\mathscr{C}_R$ is the category A-Mod/Mod-A of left/right A-modules, $\mathscr{C}_0 = \text{Ab}$ is the category of abelian groups, 1 is $\mathbb{Z}$, $\otimes = \otimes_A$ is the ordinary tensor product of modules, $\mathcal{S}_L/\mathcal{S}_R$ is the category of finitely generated projective left/right A-modules, and $(-)^*$ is the functor $\operatorname{Hom}_A(-, {}_AA_A)$.

(2) A is a graded ring, $\mathscr{C}_L/\mathscr{C}_R$ is the category A-GrMod/GrMod-A of left/right graded A-modules, $\mathscr{C}_0$ is $\mathbb{Z}$-GrMod, 1 is $\mathbb{Z}$, $\otimes = \otimes_A$ is the ususal tensor product of graded modules, $\mathcal{S}_L/\mathcal{S}_R$ is the category of finitely generated free graded left/right A-modules (that is, finite direct sums of shifts of A), and $(-)^*$ is the functor $\mathrm{Hom}_A(-, {}_AA_A)$.

(3) A is a ring, $\mathscr{C}_L/\mathscr{C}_R$ is the category Ch(A-Mod)/Ch(Mod-A) of chain complexes of left/right A-modules, $\mathscr{C}_0$ is Ch(Ab), 1 is $\mathbb{Z}$ (viewed as a complex concentrated in degree zero), $\otimes = \otimes_A$ is the total tensor product of chain complexes, $\mathcal{S}_L/\mathcal{S}_R$ is the category of bounded chain complexes of finitely generated projective left/right A-modules (these are often called *perfect complexes*), and $(-)^*$ is the functor $\mathrm{Hom}_A(-, {}_AA_A)$.

(4) A is a DGA, $\mathscr{C}_L/\mathscr{C}_R$ is the category A-DGMod/DGMod-A of left/right DG A-modules, $\mathscr{C}_0$ is Ch(Ab), 1 is $\mathbb{Z}$ (as in (3)), $\otimes = \otimes_A$ is the ususal tensor product of DG-modules, $\mathcal{S}_L/\mathcal{S}_R$ is the category of finitely generated semi-free left/right DG A-modules (that is, finite extensions of shifts of A), and $(-)^*$ is the functor $\mathrm{Hom}_A(-, {}_AA_A)$.

(5) Let $(\mathscr{C}_0, \otimes_1, 1, [-,-])$ be any closed symmetric monoidal abelian category where 1 is finitely presented. Then one can take $\mathscr{C}_L = \mathscr{C}_0 = \mathscr{C}_R$ and $\otimes = \otimes_1$. Moreover, $\mathcal{S}_L = \mathcal{S}_R$ could be the subcategory of *dualizable* objects in $\mathscr{C}_0$ (see 4.1) and $(-)^* = [-, 1]$.

These examples are all special cases of the "concrete module categories" studied in Section 4, and further in Section 5. A special case of (4) is where $\mathscr{C}_0 = \mathrm{QCoh}(X)$ is the category of quasi-coherent sheaves on a sufficiently nice scheme X and where $\mathcal{S}_L = \mathcal{S}_R$ is the category of locally free sheaves of finite rank; see 5.5 for details.

(6) $\mathcal{X}$ is an additive category, $\mathscr{C}_L/\mathscr{C}_R$ is the category $[\mathcal{X}, \mathrm{Ab}]/[\mathcal{X}^{\mathrm{op}}, \mathrm{Ab}]$ of covariant/contravariant additive functors from $\mathcal{X}$ to Ab, $\mathscr{C}_0$ is Ab, 1 is $\mathbb{Z}$, $\otimes = \otimes_{\mathcal{X}}$ is the tensor product from Oberst and Röhrl [25], $\mathcal{S}_L/\mathcal{S}_R$ is the category of representable covariant/contravariant functors, and the functor $(-)^*$ maps $\mathcal{X}(x, -)$ to $\mathcal{X}(-, x)$ and vice versa ($x \in \mathcal{X}$). See 5.6 for details.

Recall that to simplify notation we often write $\mathscr{C}$ for either $\mathscr{C}_L$ or $\mathscr{C}_R$ and $\mathcal{S}$ for either $\mathcal{S}_L$ and $\mathcal{S}_R$ (see Setup 1). In order to describe $\varinjlim \mathcal{S}$, we define from $\mathcal{S}$ three new classes of objects in $\mathscr{C}$.

Definition 6. Let $\mathscr{C}$ and $\mathcal{S}$ be as in Setup 1. Let $(\mathcal{P}, \mathcal{E})$ be the cotorsion pair in $\mathscr{C}$ generated by $\mathcal{S}$. By Proposition 2 this cotorsion pair is complete as $\mathcal{S}$ is a set.

Objects in $\mathcal{P}$ are called *semi-projective* and objects in $\mathcal{E}$ are called *acyclic* (with respect to $\mathcal{S}$). An object $M \in \mathscr{C}_L$ is called *(tensor-)flat* if the functor $- \otimes M$ is exact. A functor $F\colon \mathscr{C} \to \mathscr{C}_0$ *preserves acyclicity* if $F(\mathcal{E}) \subseteq 1^{\perp}$. Finally we say that an object $M \in \mathscr{C}_L$ is *semi-flat* if M is flat and $- \otimes M$ preserves acyclicity.

When necessary we shall use the more elaborate notation $(\mathcal{P}_L, \mathcal{E}_L)$ for the cotorsion pair in $\mathscr{C}_L$ generated by $\mathcal{S}_L$ and similarly for $(\mathcal{P}_R, \mathcal{E}_R)$.

Example 2. We immediately see that if $1 \in \mathscr{C}_0$ is projective, then semi-flat is the same as flat. This is for instance the case in A-Mod, A-GrMod and $[\mathcal{X}, \mathrm{Ab}]$ (see (1), (2), and (6) in Example 1), where every object is acyclic, and semi-projective is the same as projective. In $\mathrm{Ch}(A) = \mathrm{Ch}(A\text{-Mod})$ and in A-DGMod (see (3) and (4) in Example 1) this is not the case, and the notions acyclic, semi-projective and semi-flat agree with the usual ones found in e.g. [2]. More on this and other examples after the main theorem.

We are now ready for the main lemma. The proof is modelled on [5, Thm 1.1] which is modelled on [22, Lem 1.1]. We try to use the same notation.

Lemma 4. *With the notation of Setup 1, let $M \in \mathscr{C}_L$ be an object such that $-\otimes M$ is left exact and $\mathscr{C}_0(1, \varphi \otimes M)$ is epi whenever φ is epi in $\mathscr{C}_R$ with $\ker\varphi \in \mathcal{E}_R$. Then $M \in \varinjlim \mathcal{S}_L$.*

Proof. By Lemma 1 we need to fill in the dashed part of the following diagram

$$\begin{array}{ccc} P & \xrightarrow{u} & M \\ & \overset{v}{\searrow} & \uparrow w \\ & & L \end{array}$$

for some $L \in \mathcal{S}_L$, where u is given with P finitely presented. So let u be given.

By Proposition 1, P has a presentation

$$L_1 \xrightarrow{f} L_0 \xrightarrow{g} P \longrightarrow 0$$

with $L_1, L_0 \in \mathcal{S}_L$. We have an exact sequence

$$0 \longrightarrow K \xrightarrow{k} L_0^* \xrightarrow{f^*} L_1^* ,$$

which, since $-\otimes M$ and $\mathscr{C}_0(1, -)$ are left exact, gives an exact sequence

$$0 \longrightarrow \mathscr{C}_0(1, K \otimes M) \xrightarrow{k_*} \mathscr{C}_L(L_0, M) \xrightarrow{f_*} \mathscr{C}_L(L_1, M)$$

where we have used $\mathscr{C}_0(1, L_j^* \otimes M) \cong \mathscr{C}_L(L_j, M)$ for $j = 0, 1$.

By completeness of the cotortion pair $(\mathcal{P}_R, \mathcal{E}_R)$, the object K has a presentation

$$0 \longrightarrow E \longrightarrow L' \xrightarrow{\varphi} K \longrightarrow 0$$

with $L' \in \mathcal{P}_R$ and $E \in \mathcal{E}_R$. By assumption, $\varphi_* = \mathscr{C}_0(1, \varphi \otimes M)$ is epi, so we get an exact sequence

$$\mathscr{C}_0(1, L' \otimes M) \xrightarrow{k_*\varphi_*} \mathscr{C}_L(L_0, M) \xrightarrow{f_*} \mathscr{C}_L(L_1, M).$$

Now since $f_*(ug) = ugf = 0$, we have some $w' \colon 1 \to L' \otimes M$ such that $(k\varphi)_*(w') = ug$. By Proposition 3 we can realize L' as a direct limit $\varinjlim L_i^*$, with $L_i \in \mathcal{S}_L$. This means that we have

$$L' \otimes M \cong (\varinjlim L_i^*) \otimes M \cong \varinjlim (L_i^* \otimes M),$$

as $\otimes$ is right continuous. Since 1 is finitely presented, w' factors as

$$1 \xrightarrow{w} L^* \otimes M \xrightarrow{\iota \otimes M} L' \otimes M$$

for some $L \in \mathcal{S}_L$ and $w \in \mathscr{C}_L(L, M) \cong \mathscr{C}_0(1, L^* \otimes M)$. By the assumed duality between $\mathcal{S}_L$ and $\mathcal{S}_R$ there exists $v' \colon L_0 \to L$ such that $v'^* = k\varphi\iota$. We now have the commutative diagram

$$\begin{array}{ccccc} \mathscr{C}_L(L, M) & & & & \\ \downarrow \iota_* & \overset{v'_*}{\searrow} & & & \\ \mathscr{C}_0(1, L' \otimes M) & \xrightarrow{k_*\varphi_*} & \mathscr{C}_L(L_0, M) & \xrightarrow{f_*} & \mathscr{C}_L(L_1, M) \end{array}$$

where $wv' = v'_*(w) = k_*(\varphi_*(w')) = ug$. This gives us the commutative diagram

$$\begin{array}{ccccccc} L_1 & \xrightarrow{f} & L_0 & \xrightarrow{g} & P & \longrightarrow & 0 \\ & \overset{0}{\searrow} & & \overset{v'}{\searrow} & \downarrow v & \overset{u}{\searrow} & \\ & & & & L & \xrightarrow{w} & M \end{array}$$

where $v'f = 0$ since $f^* v'^* = f^* k\varphi\iota = 0\varphi\iota = 0$. Thus v' factors through g by some v as g is the cokernel of f. It remains to note that $wv = u$, as desired. □

Remark 4. The main difference between this proof and the proof in [5] is that all the relevant identities have been formalized instead of based on calculations with elements, in particular, the use of the generalized Hill Lemma instead of element considerations to find the right $\mathcal{S}$-subobject of a semi-projective object.

Lemma 4 will allow us to prove that every semi-flat object belongs to the direct limit closure of $\mathcal{S}$ (see Theorem 1 below). For the converse statement, we need the following setup.

Setup 2. With the notation of Setup 1 assume further that 1 is FP_2, i.e. $\mathrm{Ext}_{\mathscr{C}_0}(1,-)$ respects direct limits, and that for any $S \in \mathcal{S}_L$ we have that $- \otimes S$ is exact and there are natural isomorphisms:

$$\mathrm{Ext}_{\mathscr{C}_0}(1, S^* \otimes -) \cong \mathrm{Ext}_{\mathscr{C}_L}(S, -) \quad \text{and}$$
$$\mathrm{Ext}_{\mathscr{C}_0}(1, - \otimes S) \cong \mathrm{Ext}_{\mathscr{C}_R}(S^*, -)\,.$$

By the duality between $\mathcal{S}_L$ and $\mathcal{S}_R$, similar conditions hold for $S \in \mathcal{S}_R$. (Note that the isomorphisms above are the "Ext versions" of the isomorphisms from Setup 1.)

As in Remark 3 one sees that in the setting of Setup 2 every $S \in \mathcal{S}$ is FP_2.

We can now link the direct limit closure to semi-flatness (from Definition 6).

Theorem 1. *Let $\mathscr{C}$ and $\mathcal{S}$ be as in Setup 1. If $M \in \mathscr{C}$ is semi-flat, then $M \in \varinjlim \mathcal{S}$. Conversely, if $\mathscr{C}$ and $\mathcal{S}$ satisfy the conditions of Setup 2, then every $M \in \varinjlim \mathcal{S}$ is semi-flat.*

Proof. To use Lemma 4, we just need to see, that if $M \in \mathscr{C}_L$ is semi-flat, then $\mathscr{C}_0(1, \varphi \otimes_A M)$ is epi whenever φ is epi and $\ker \varphi$ is acyclic. This is clear, since if

$$0 \longrightarrow E \longrightarrow A \xrightarrow{\varphi} B \longrightarrow 0$$

is exact and E is acyclic, then

$$0 \longrightarrow E \otimes M \longrightarrow A \otimes M \xrightarrow{\varphi \otimes M} B \otimes M \longrightarrow 0$$

is exact and $\mathrm{Ext}_{\mathscr{C}_0}(1, E \otimes M) = 0$. But this implies that $\mathscr{C}_0(1, \varphi \otimes M)$ is epi.

For the other direction we show that every $S \in \mathcal{S}_L$ is semi-flat and that the class of semi-flat objects in $\mathscr{C}_L$ is closed under direct limits. First observe that if $E \in \mathcal{E}_R = \mathcal{S}_R^{\perp}$ and $S \in \mathcal{S}_L$ then

$$\mathrm{Ext}_{\mathscr{C}_0}(1, E \otimes S) \cong \mathrm{Ext}_{\mathscr{C}_R}(S^*, E) = 0\,,$$

so $- \otimes S$ preserves acyclicity, and since $- \otimes S$ is assumed to be exact, S is semi-flat. Now if $M_i \in \mathscr{C}_L$ is a direct system of semi-flat objects and $M = \varinjlim M_i$, then $- \otimes M$ is exact as $\otimes$ is right continuous and $\varinjlim(-)$ is exact. It also preserves acyclicity, as

$$\mathrm{Ext}_{\mathscr{C}_0}(1, E \otimes \varinjlim M_i) \cong \mathrm{Ext}_{\mathscr{C}_0}(1, \varinjlim(E \otimes M_i)) \cong \varinjlim \mathrm{Ext}_{\mathscr{C}_0}(1, E \otimes M_i) = 0$$

as $\otimes$ is right continuous, $\mathrm{Ext}_{\mathscr{C}_0}(1,-)$ respects direct limits and $\mathrm{Ext}_{\mathscr{C}_0}(1, E \otimes M_i) = 0$. Hence M is semi-flat. □

Corollary 1. *Let $\mathscr{C}$ and $\mathcal{S}$ be as in Setup 1 and assume further that that $1 \in \mathscr{C}_0$ is projective and that every $S \in \mathcal{S}$ is (tensor-)flat. Then the direct limit closure of $\mathcal{S}$ is the class of (tensor-)flat objects in $\mathscr{C}$.*

Proof. As in Example 2 semi-flat just means flat if 1 is projective, so by Theorem 1 every flat object in $\mathscr{C}$ is in the direct limit closure of $\mathcal{S}$. On the other hand any $S \in \varinjlim \mathcal{S}$ is flat as this is preserved by direct limits as in the proof of Theorem 1. □

As mentioned in the Introduction, we will now see how Theorem 1 recovers Govorov and Lazard's original theorem for modules, the theorem by Christensen and Holm for complexes of modules, the theorem by Oberst and Röhrl for functor categories, and how it gives new results for graded modules, differential graded modules, and quasi-coherent sheaves.

Most of these examples are built of categories of left/right objects for some monoid in a symmetric monoidal category. So in the next section we will explain this construction with a new definition of *dualizable* objects in such categories and show in what cases they satisfy Setup 1 and 2. Then we will go in depth with the more concrete examples, calculating the different classes of objects.

4. Concrete module categories

Setup 3. The details of this setup can be found in Pareigis [26]. Consider any closed symmetric monoidal abelian category $\mathscr{C}_0 = (\mathscr{C}_0, \otimes_1, [-,-], 1)$. A *monoid* (or a *ring object*) in $\mathscr{C}_0$ is an object, A, together with an associative multiplication $A \otimes_1 A \to A$ with a unit $1 \to A$. We can then consider the category ${}_A\mathscr{C}$ of *left A-modules* whose objects are objects $X \in \mathscr{C}_0$ equipped with a left A-multiplication $A \otimes_1 X \to X$ respecting the multiplication of A on the left and the unit. The morphisms are morphims in $\mathscr{C}_0$ respecting this left A-multiplication. We can also consider the category $\mathscr{C}_A$ of *right A-modules* and the category ${}_A\mathscr{C}_A$ of (A, A)-*bimodules*, that is, simultaneously left and right A-modules with compatible actions.

We can then construct a functor $\otimes_A \colon \mathscr{C}_A \times {}_A\mathscr{C} \to \mathscr{C}_0$ as a coequalizer:

$$Y \otimes_1 A \otimes_1 X \rightrightarrows Y \otimes_1 X \longrightarrow Y \otimes_A X\ .$$

And we get induced functors ${}_A\mathscr{C}_A \times {}_A\mathscr{C} \to {}_A\mathscr{C}$ and $\mathscr{C}_A \times {}_A\mathscr{C}_A \to \mathscr{C}_A$ with $A \otimes_A X \cong X$ in ${}_A\mathscr{C}$ and $Y \otimes_A A \cong Y$ in $\mathscr{C}_A$.

We can also construct ${}_A[-,-] \colon {}_A\mathscr{C} \times {}_A\mathscr{C} \to \mathscr{C}_0$ as an equalizer

$${}_A[X, X'] \longrightarrow [X, X'] \rightrightarrows [A \otimes_1 X, X']$$

and similarly for $[-,-]_A \colon \mathscr{C}_A \times \mathscr{C}_A \to \mathscr{C}_0$. Again we get induced functors ${}_A[-,-] \colon {}_A\mathscr{C} \times {}_A\mathscr{C}_A \to \mathscr{C}_A$ and $[-,-]_A \colon \mathscr{C}_A \times {}_A\mathscr{C}_A \to {}_A\mathscr{C}$.

There are natural isomorphisms:

$$\begin{aligned} {}_A\mathscr{C}(X \otimes_1 Z, X') &\cong \mathscr{C}_0(Z, {}_A[X, X'])\,, \\ \mathscr{C}_A(Z \otimes_1 Y, Y') &\cong \mathscr{C}_0(Z, [Y, Y']_A)\,, \text{ and} \\ {}_A[X \otimes_1 Z, X'] &\cong [Z, {}_A[X, X']]\,. \end{aligned}$$

That is, $X \otimes_1 -$ and ${}_A[X, -]$ (as well as $- \otimes_1 Y$ and $[Y, -]_A$) are adjoints. Similarly, $- \otimes_A X$ and $[X, -]$ (as well as $Y \otimes_A -$ and $[Y, -]$) are adjoints. We denote the unit and the counit of the adjunctions by η and ε. As $A \in {}_A\mathscr{C}_A$, we can define functors $(-)^* = {}_A[-, A] \colon {}_A\mathscr{C} \to \mathscr{C}_A$ and $(-)^* = [-, A]_A \colon \mathscr{C}_A \to {}_A\mathscr{C}$ with $A^* \cong A$, where on one side, A is regarded as a left A-module, and on the other side, A is regarded as a right A-module. Also notice that ${}_1\mathscr{C} \cong \mathscr{C}_0 \cong \mathscr{C}_1$. Again all the details are in [26].

In accordance with our convention from Setup 1, we often write $\mathscr{C}$ for either ${}_A\mathscr{C}$ or $\mathscr{C}_A$.

The forgetful functor from ${}_A\mathscr{C} \to \mathscr{C}_0$ creates limits, colimits and isomorphisms [26, 2.4] and thus we get:

Proposition 4. *If $\mathscr{C}_0$ is Grothendieck generated by a collection $\{X\}$ of (finitely presented) objects, then ${}_A\mathscr{C}$ is Grothendieck generated by the collection $\{A \otimes_1 X\}$ of (finitely presented) objects.*

Proof. We only prove the assertion about the generators (see Definition 1). Assume that $\mathscr{C}_0$ is generated by $\{X\}$. Let $Y \to Y'$ be a non-zero morphism in ${}_A\mathscr{C}$. Then $Y \to Y'$ is also non-zero in $\mathscr{C}_0$, so we can find some X in the collection $\{X\}$ and a morphism $f\colon X \to Y$ such that $X \to Y \to Y'$ is non-zero in $\mathscr{C}_0$. Now the morphism $X \to A \otimes_1 X \to A \otimes_1 Y \to Y \to Y'$ is non-zero as $X \to A \otimes_1 X \to A \otimes_1 Y \to Y$ is equal to f, and hence $A \otimes_1 X \to A \otimes_1 Y \to Y \to Y'$ must be non-zero as well. Thus the collection $\{A \otimes_1 X\}$ generates ${}_A\mathscr{C}$. If X is finitely presented in $\mathscr{C}_0$, then $A \otimes_1 X$ is finitely presneted in ${}_A\mathscr{C}$ since ${}_A\mathscr{C}(A \otimes_1 X, -) \cong \mathscr{C}_0(X, {}_A[A, -])$ and the forgetful functor ${}_A[A, -]\colon {}_A\mathscr{C} \to \mathscr{C}_0$ preserves colimits. □

4.1. **Dualizable objects.** In [23, III§1] Lewis and May define *finite* objects in a closed symmetric monoidal category. Such objects are called *(strongly) dualizable* in Hovey, Palmieri, and Strickland [19]. We extend this notion to categories of left/right modules over a monoid in a closed symmetric monoidal category by the following definition.

First, ε (introduced above) induces a map

$$\mathscr{C}_0(Z, X^* \otimes_A X') \longrightarrow {}_A\mathscr{C}(X \otimes_1 Z, X'),$$

for any $X, X' \in {}_A\mathscr{C}$ and $Z \in \mathscr{C}_0$, by $X \otimes_1 Z \longrightarrow X \otimes_1 X^* \otimes_A X' \xrightarrow{\varepsilon \otimes_A X'} X'$.

Next, ε induces a morphism

$$\nu\colon {}_A[X, Z] \otimes_A X' \longrightarrow {}_A[X, Z \otimes_A X'],$$

for any $X, X' \in {}_A\mathscr{C}$ and $Z \in {}_A\mathscr{C}_A$, by the adjoint of $X \otimes_1 {}_A[X, Z] \otimes_A X' \xrightarrow{\varepsilon \otimes_A X'} Z \otimes_A X'$. We can now give the following:

Definition 7. An object $X \in {}_A\mathscr{C}$ is said to be *dualizable* if there exists a morphism $\eta'\colon 1 \to X^* \otimes_A X$ in $\mathscr{C}_0$ such that the following diagram commutes:

$$\begin{array}{ccc} 1 & \xrightarrow{\eta'} & X^* \otimes_A X \\ {\scriptstyle \eta}\downarrow & \swarrow{\scriptstyle \nu} & \\ {}_A[X, X] & & \end{array}$$

Similarly, one defines what it means for an object in $\mathscr{C}_A$ to be dualizable.

Note that $A \in {}_A\mathscr{C}$ and $A \in \mathscr{C}_A$ are always dualizable.

Many equivalent descriptions of dualizable objects can be given, and we give several in the next lemma.

Lemma 5. *For $X \in {}_A\mathscr{C}$ the following conditions are equivalent:*

(1) *There exists a morphism $\eta'\colon 1 \to X^* \otimes_A X$ in $\mathscr{C}_0$ making the following diagram commute:*

$$\begin{array}{ccc} X & \xrightarrow{X \otimes_1 \eta'} & X \otimes_1 X^* \otimes_A X \\ & {\scriptstyle =}\searrow \quad \swarrow{\scriptstyle \varepsilon \otimes_A X} & \\ & X & \end{array}$$

(2) $\mathscr{C}_0(1, X^* \otimes_A X) \xrightarrow{\cong} {}_A\mathscr{C}(X, X)$ *induced by* ε.
(3) $\mathscr{C}_0(1, X^* \otimes_A X') \xrightarrow{\cong} {}_A\mathscr{C}(X, X')$ *induced by* ε *for all* $X' \in {}_A\mathscr{C}$.
(4) $\mathscr{C}_0(Z, X^* \otimes_A X') \xrightarrow{\cong} {}_A\mathscr{C}(X \otimes_1 Z, X')$ *induced by* ε *for all* $X' \in {}_A\mathscr{C}$ *and* $Z \in \mathscr{C}_0$.
(5) $\mathscr{C}_0(Z, Y \otimes_A X') \xrightarrow{\cong} {}_A\mathscr{C}(X \otimes_1 Z, X')$ *for some* $Y \in \mathscr{C}_A$ *and all* $X' \in {}_A\mathscr{C}, Z \in \mathscr{C}_0$.
(6) X *is dualizable.*
(7) $\nu\colon X^* \otimes_A X \xrightarrow{\cong} {}_A[X, X]$.
(8) $\nu\colon {}_A[X, Z] \otimes_A X' \xrightarrow{\cong} {}_A[X, Z \otimes_A X']$ *for all* $X' \in {}_A\mathscr{C}$ *and* $Z \in {}_A\mathscr{C}_A$.

(9) $\nu\colon {}_A[X', Z] \otimes_A X \xrightarrow{\cong} {}_A[X', Z \otimes_A X]$ *for all* $X' \in {}_A\mathscr{C}$ *and* $Z \in {}_A\mathscr{C}_A$.

Proof. Clearly (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1), and (1) $\Rightarrow$ (4) as η' from (1) induces a map ${}_A\mathscr{C}(X \otimes_1 Z, X') \to \mathscr{C}_0(Z, X^* \otimes_A X')$ by $Z \to X^* \otimes_A X \otimes_1 Z \to X^* \otimes_A X'$, and the diagram from (1) precisely says that it is an inverse to the map induced by ε. Clearly, either of the conditions (8) and (9) imply (7), and (7) $\Rightarrow$ (6). The implications (6) $\Rightarrow$ (8) and (6) $\Rightarrow$ (9) can be proved as in [23, III Prop. 1.3(ii)]. We also have (1) $\Leftrightarrow$ (6) as the diagrams in question are adjoint, so we are left with noting that (4) $\Rightarrow$ (5) (trivial) and that (5) $\Rightarrow$ (6) can be proved as in [23, III Thm. 1.6]. □

Remark 5. We notice that Lemma 5 (5) makes no mention of the functor $[-,-]$ and thus this condition can be used to define dualizable objects in, for example, symmetric monoidal categories that are not closed. In this case, Y is a "dual" of X. We chose a definition with a fixed dual object, $X^* = {}_A[X, A]$, because this emphasizes the canonical and thereby functorial choice of a dual object.

Next we show three lemmas about closure properties for the class of dualizable objects.

Lemma 6. $(-)^*$ *induces a duality between the categories of dualizable objects in* ${}_A\mathscr{C}$ *and dualizable objects in* $\mathscr{C}_A$. *In particular, if* X *is dualizable, then so is* X^* *and the adjoint of* ε *gives an isomorphism* $X \xrightarrow{\cong} X^{**}$.

Proof. As in [23, Prop. 1.3(i)]. □

Lemma 7. *Dualizable objects are closed under extensions and direct summands.*

Proof. The closure under direct summands follows directly from Lemma 5 (3).

So assume that

$$0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow 0$$

is exact and X_1, X_3 are dualizable (in ${}_A\mathscr{C}$). Then we have the following commutative diagram in $\mathscr{C}_0$ with exact rows

$$\begin{array}{ccccccccc}
 & & X_2^* \otimes_A X_1 & \longrightarrow & X_2^* \otimes_A X_2 & \longrightarrow & X_2^* \otimes_A X_3 & \longrightarrow & 0 \\
 & & \simeq\downarrow & & \downarrow & & \simeq\downarrow & & \\
0 & \longrightarrow & {}_A[X_2, X_1] & \longrightarrow & {}_A[X_2, X_2] & \longrightarrow & {}_A[X_2, X_3]\,, & &
\end{array}$$

where the outer vertical morphisms are isomorphisms by Lemma 5 (9), so the middle morphism is an isomorphism by the snake lemma. Hence X_2 is dualizable by Lemma 5 (7). □

Lemma 8. *If* S *is dualizable in* $\mathscr{C}_0$, *then*

$$(X \otimes_1 S)^* \cong S^* \otimes_1 X^*$$

for any $X \in {}_A\mathscr{C}$. *If* $X \in {}_A\mathscr{C}$ *is dualizable then so is* $X \otimes_1 S \in {}_A\mathscr{C}$. *In particular,* $A \otimes_1 S \in {}_A\mathscr{C}$ *and* $(A \otimes_1 S)^* \cong S^* \otimes_1 A \in \mathscr{C}_A$ *are dualizable if* $S \in \mathscr{C}_0$ *is dualizable.*

Proof. If $S \in \mathscr{C}_0$ is dualizable, then we have

$$(X \otimes_1 S)^* = {}_A[X \otimes_1 S, A] \cong [S, {}_A[X, A]] \cong [S, 1 \otimes_1 X^*] \cong [S, 1] \otimes_1 X^* = S^* \otimes_1 X^*.$$

When X is also dualizable we have

$$\mathscr{C}_0(1, (X \otimes_1 S)^* \otimes_A -) \cong \mathscr{C}_0(1, S^* \otimes_1 X^* \otimes_A -) \cong \mathscr{C}_0(S, X^* \otimes_A -) \cong {}_A\mathscr{C}(X \otimes_1 S, -)$$

on ${}_A\mathscr{C}$, and hence $X \otimes_1 S$ is dualizable in ${}_A\mathscr{C}$ by Lemma 5 (3). □

We now have a large supply of categories satisfying Setup 1 and 2

Theorem 2. *Let A be a monoid in a closed symmetric monoidal Grothendieck category $(\mathscr{C}_0, \otimes_1, [-,-], 1)$ where 1 is finitely presented. Assume that $\mathscr{C}_0$ is generated by a set $\mathcal{S}$ of dualizable objects such that $\mathcal{S}^*$ also generates $\mathscr{C}_0$ (e.g. if $\mathcal{S} = \mathcal{S}^*$). Assume furthermore that ${}_A\mathcal{S}$ is a collection of dualizable objects in ${}_A\mathscr{C}$ which is closed under extensions and contains $A \otimes_1 \mathcal{S}$ (e.g. ${}_A\mathcal{S}$ could be the collection of all dualizable objects in ${}_A\mathscr{C}$; see Lemmas 7 and 8).*

(1) *Under the assumptions above, the data $\mathscr{C}_L := {}_A\mathscr{C}$, $\mathscr{C}_R := \mathscr{C}_A$, $\otimes := \otimes_A$, $(-)^* := {}_A[-, A]$, $\mathcal{S}_L := {}_A\mathcal{S}$ and $\mathcal{S}_R := ({}_A\mathcal{S})^*$ satisfy Setup 1.*
In particular, Theorem 1 yields that every semi-flat object in ${}_A\mathscr{C}$, respectively, in $\mathscr{C}_A$, belongs to $\varinjlim {}_A\mathcal{S}$, respectively, to $\varinjlim \mathcal{S}_A$.

(2) *If, in addition, 1 is FP_2, then Setup 2 holds as well.*
In particular, Theorem 1 yields that the class of semi-flat objects in ${}_A\mathscr{C}$, respectively, in $\mathscr{C}_A$, is precisely $\varinjlim {}_A\mathcal{S}$, respectively, $\varinjlim \mathcal{S}_A$.

(3) *If, in addition, 1 is projective, then $\varinjlim {}_A\mathcal{S}$, respectively, $\varinjlim \mathcal{S}_A$, is precisely the (tensor-)flat objects in ${}_A\mathscr{C}$, respectively, in $\mathscr{C}_A$.*

Proof. (1): First note that since $1 \in \mathscr{C}_0$ is finitely presented, so is every dualizable object. Indeed, for e.g. $S \in {}_A\mathscr{C}$ one has ${}_A\mathscr{C}(S, -) \cong \mathscr{C}_0(1, S^* \otimes_A -)$; cf. Remark 3. Proposition 4 shows that ${}_A\mathscr{C}$ is Grothendieck generated by the set $A \otimes_1 \mathcal{S} \subseteq {}_A\mathcal{S}$. The objects in the set $A \otimes_1 \mathcal{S}$ are dualizable, cf. Lemma 8, and hence finitely presented by the observation above. Consequently, ${}_A\mathscr{C}$ is a locally finitely presented Grothendieck category, and $\mathrm{fp}({}_A\mathscr{C})$ is small by Remark 1; hence ${}_A\mathcal{S} \subseteq \mathrm{fp}({}_A\mathscr{C})$ is small. Similarly, $\mathscr{C}_A$ is Grothendieck generated by the set $\mathcal{S}^* \otimes_1 A = (A \otimes_1 \mathcal{S})^* \subseteq ({}_A\mathcal{S})^*$; see Lemma 8. And as ${}_A\mathcal{S}$ is small, so is $({}_A\mathcal{S})^*$. By Lemma 6 the class $({}_A\mathcal{S})^*$ consists of dualizable objects and $(-)^*$ yields a duality between ${}_A\mathcal{S}$ and $({}_A\mathcal{S})^*$. Since ${}_A\mathcal{S}$ is closed under extensions, the same is true for $({}_A\mathcal{S})^*$ (by the duality). The natural isomorphisms in Setup 1 hold by Lemma 5 (3). It remains to note that $\otimes_A$ is a right continuous bifunctor, as it is a left adjoint in both variables.

(2): Assume that 1 is FP_2. Every $S \in {}_A\mathcal{S}$ is dualizable, so the functor $- \otimes_A S$ is exact. Thus, to establish Setup 2 it remains to prove the two natural isomorphisms herein. We only prove the second of these, i.e. $\mathrm{Ext}_{\mathscr{C}_0}(1, - \otimes_A S) \cong \mathrm{Ext}_{\mathscr{C}_A}(S^*, -)$ for $S \in {}_A\mathcal{S}$. The first one is proved similarly. To this end, we apply Lemma 2 to the adjunction $S^* \otimes_1 - \colon \mathscr{C}_0 \leftrightarrows \mathscr{C}_A : - \otimes_A S$ from Lemma 5 (4). The right adjoint $- \otimes_A S$ is clearly exact as S is dualizable in ${}_A\mathscr{C}$. It remains to show that the left adjoint functor $S^* \otimes_1 -$ leaves every short exact sequence $0 \to D \to E \to 1 \to 0$ (ending in 1) exact. To see this, first note that the category $\mathscr{C}_0$ has *enough $\otimes_1$-flats*, that is, for every object $X \in \mathscr{C}_0$ there exists an epimorphism $F \twoheadrightarrow X$ in $\mathscr{C}_0$ where F is $\otimes_1$-flat. Indeed, this follows from Stenström [31, IV.6 Prop. 6.2] as $\mathscr{C}_0$ has coproducts and is generated by a set of $\otimes_1$-flat (even dualizable) objects. Consequently, S^* has a $\otimes_1$-flat resolution $F_\bullet = \cdots \to F_1 \to F_0 \to 0$ in $\mathscr{C}_0$. Every short exact sequence $0 \to D \to E \to 1 \to 0$ in $\mathscr{C}_0$ induces a short exact sequence $0 \to F_\bullet \otimes_1 D \to F_\bullet \otimes_1 E \to F_\bullet \to 0$ of chain complexes in $\mathscr{C}_0$ which, in turn, yields a long exact sequence in homology,

$$\cdots \to H_1(F_\bullet) \to H_0(F_\bullet \otimes_1 D) \to H_0(F_\bullet \otimes_1 E) \to H_0(F_\bullet) \to H_{-1}(F_\bullet \otimes_1 D) \to \cdots .$$

Evidently, $H_1(F_\bullet) = 0 = H_{-1}(F_\bullet \otimes_1 D)$. As the functor $- \otimes_1 X$ is right exact we get $H_0(F_\bullet \otimes_1 X) \cong S^* \otimes_1 X$ for all $X \in \mathscr{C}_0$, and so $0 \to S^* \otimes_1 D \to S^* \otimes_1 E \to S^* \to 0$ is exact, as desired.

(3) Immediate from Corollary 1. ☐

For closed symmetric monoidal Grothendieck categories we get the following.

Corollary 2. *Let $(\mathscr{C}_0, \otimes_1, [-,-], 1)$ be a closed symmetric monoidal Grothendieck category where 1 is finitely presented. Assume that $\mathscr{C}_0$ is generated by the set $\mathcal{S}$ of dualizable objects. Then the following hold:*

(1) *Every semi-flat object in $\mathscr{C}$ belongs to $\varinjlim \mathcal{S}$.*
(2) *If 1 is FP_2, then the class of semi-flat objects in $\mathscr{C}$ is precisely $\varinjlim \mathcal{S}$.*
(3) *If 1 is projective, then $\varinjlim \mathcal{S}$ is precisely the (tensor-)flat objects in $\mathscr{C}$.*

Remark 6. Consider the situation from Theorem 2. If $1 \in \mathscr{C}_0$ is projective, then all objects in ${}_A\mathcal{S}$ are projective. Indeed, consider any $S \in {}_A\mathcal{S}$. By Lemma 5 (4) we have the adjunction $S \otimes_1 - \colon \mathscr{C}_0 \leftrightarrows {}_A\mathscr{C} : S^* \otimes_A -$, and since the right adjoint functor $S^* \otimes_A -$ is exact, the left adjoint functor $S \otimes_1 -$ preserves projective objects. Hence, if $1 \in \mathscr{C}_0$ is projective, then so is $S \otimes_1 1 \cong S \in {}_A\mathcal{S}$.

Thus, in the case where $1 \in \mathscr{C}_0$ is projective, the cotorsion pair $(\mathcal{P}, \mathcal{E})$ in ${}_A\mathscr{C}$ generated by ${}_A\mathcal{S}$ is the trivial cotorsion where $\mathcal{P}$ is the class of all projective objects and $\mathcal{E} = {}_A\mathscr{C}$ (cf. Definition 6). Similarly for $\mathcal{S}_A$ and $\mathscr{C}_A$.

5. Examples

In this final section, we return to the examples from Example 1 and to the results from the literature mentioned in the Introduction.

5.1. ***A*-Mod.** $\mathscr{C}_0 = \mathrm{Ab}$ is a Grothendieck category generated by $1 = \mathbb{Z}$, which is finitely presented and projective, and ${}_A\mathscr{C}$ is just A-Mod. The condition in Definition 7 is equivalent to the existence of a finite number of elements $f_i \in X^*$ and $x_i \in X$ such that $x = \sum_i f_i(x)x_i$ for any $x \in X$. By the Dual Basis Theorem [24, Chap. 2.3], this is precisely the finitely generated projective R-modules. Also the finitely generated free modules are closed under extensions and contains $R \otimes_{\mathbb{Z}} \mathbb{Z} \cong R$, so by Theorem 2(3) we get the original theorem of Lazard and Govorov:

Corollary 3. *Over any ring, the flat modules are the direct limit closure of the finitely generated projective (or free) modules.* □

5.2. ***A*-GrMod.** $\mathscr{C}_0 = \mathbb{Z}$-GrMod is a Grothendieck category where $1 = \mathbb{Z}$ is finitely presented and projective. The category $\mathscr{C}_0$ is generated by the set $\mathcal{S} = \{\Sigma^i 1\}_{i \in \mathbb{Z}}$, which is self-dual (that is, $\mathcal{S}^* = \mathcal{S}$) and consists of dualizable objects. Also note that ${}_A\mathscr{C}$ is just A-GrMod. A graded A-module is *finitely generated free* if it is a finite direct sums of shifts of A, and it is *finitely generated projective* if it is a direct summand of a finitely generated free graded A-module. Arguments like the ones above show that the dualizable objects in ${}_A\mathscr{C}$ are precisely the finitely generated projective graded A-modules. Thus by Theorem 2(3) we get the following version of Govorov-Lazard for graded modules (which does not seem to be available in the literature):

Corollary 4. *Over any $\mathbb{Z}$-graded ring, the flat graded modules are the direct limit closure of the finitely generated projective (or free) graded modules.* □

5.3. ***A*-DGMod.** $\mathscr{C}_0 = \mathrm{Ch}(\mathrm{Ab})$, the category of chain complexes of abelian groups, is a Grothendieck category where 1 is the complex with $\mathbb{Z}$ concentrated in degree 0. Note that 1 is finitely presented (but not projective!), as $\mathscr{C}_0(1,-) \cong Z_0(-)$ is the 0^{th} cycle functor which preserves direct limits. The category $\mathscr{C}_0$ is generated by the set $\mathcal{S} = \{\Sigma^i M(\mathrm{Id}_1)\}_{i \in \mathbb{Z}}$ (where $M(\mathrm{Id}_1)$ is the mapping cone of the identity morphism on 1), which is self-dual (i.e. $\mathcal{S}^* = \mathcal{S}$) and consists of dualizable objects.

A monoid A in $\mathscr{C}_0 = \mathrm{Ch}(\mathrm{Ab})$ is a differential graded algebra and ${}_A\mathscr{C}$ is the category A-DGMod of differential graded left A-modules. DG-modules are thus covered by Setup 3. Clearly any shift of A is dualizable, so by Lemma 7 any finite extension of shifts of A will be dualizable, and we call such modules *finitely generated semi-free*. Direct summands of those are called *finitely generated semi-projectives*. I do not know if the finitely generated semi-projective DG-modules constitute all dualizable objects in ${}_A\mathscr{C} = A$-DGMod for a general DGA. Nevertheless, it is not hard to check that both of the above mentioned classes are self-dual and closed under extensions. They also contain $A \otimes_{\mathbb{Z}} \mathcal{S}$ and thus satisfy Setup 1 by Theorem 2(1).

Actually, $1 = \mathbb{Z}$ is not just finitely presented but even FP_2, so from Theorem 2(2) we conclude that the direct limit closure of the finitely generated semi-free/semi-projective DG-modules is precisely the class of semi-flat objects in ${}_A\mathscr{C} = A$-DGMod in the abstract sense of Definition 6. Before we go further into this, let's see that our abstract notions of semi-projective, acyclic and semi-flat objects from Definition 6 agree with the usual ones. These notions originate in the treatise [2] by Avramov, Foxby, and Halperin, where several equivalent conditions are given.

Definition 8. Let A be any DGA and let ${}_A\mathscr{C} = A$-DGMod.

- A DG-module is called *acyclic* (or *exact*) if it has trivial homology.
- A DG-module, P, is called *semi-projective* (or *DG-projective*) if ${}_A\mathscr{C}(P, \psi)$ is epi, whenever ψ is epi and $\ker \psi$ has trivial homology (in other words, ψ is a surjective quasi-isomorphism).
- A DG-module, M, is called *semi-flat* (or *DG-flat*) if $- \otimes_A M$ is exact and preserves acyclicity (i.e. $E \otimes_A M$ has trivial homology whenever E has).

First we notice that:

Lemma 9. *A DG-module P is DG-projective iff $\mathrm{Ext}^1_{{}_A\mathscr{C}}(P, E) = 0$ whenever E is a DG-module with trivial homology.*

Proof. If $\mathrm{Ext}^1_{{}_A\mathscr{C}}(P, E) = 0$ and

$$0 \longrightarrow E \longrightarrow A \xrightarrow{\varphi} B \longrightarrow 0$$

is an exact sequence, then clearly ${}_A\mathscr{C}(P, \varphi)$ is epi. On the other hand, if

$$0 \longrightarrow E \longrightarrow X \xrightarrow{\varphi} P \longrightarrow 0$$

is exact and ${}_A\mathscr{C}(P, \varphi)$ is epi, then the sequence split, so $\mathrm{Ext}^1_{{}_A\mathscr{C}}(P, E) = 0$. □

Next we see that:

Lemma 10. *Let A be a DGA. For any $N \in {}_A\mathscr{C}$ we have $\mathrm{Ext}^1_{{}_A\mathscr{C}}(\Sigma A, N) = H_0(N)$.*

Proof. To compute this, we use the short exact sequence

$$0 \longrightarrow A \longrightarrow M(\mathrm{Id}_A) \longrightarrow \Sigma A \longrightarrow 0$$

where $M(\mathrm{Id}_A)$ is the mapping cone of $A \xrightarrow{\mathrm{Id}_A} A$. Since $M(\mathrm{Id}_A)$ is projective we have $\mathrm{Ext}^1_{{}_A\mathscr{C}}(M(\mathrm{Id}_A), N) = 0$, so we get an exact sequence

$${}_A\mathscr{C}(M(\mathrm{Id}_A), N) \longrightarrow {}_A\mathscr{C}(A, N) \longrightarrow \mathrm{Ext}^1_{{}_A\mathscr{C}}(\Sigma A, N) \longrightarrow 0$$

Straightforward calculations show that this sequence is isomorphic to

$$N_1 \xrightarrow{\partial_1^N} Z_0(N) \longrightarrow \mathrm{Ext}^1_{{}_A\mathscr{C}}(\Sigma A, N) \longrightarrow 0$$

where N_1 is the degree 1 part of N and ∂_1^N is the differential. Thus we get the desired isomorphim $\mathrm{Ext}^1_{{}_A\mathscr{C}}(\Sigma A, N) \cong H_0(N)$. □

Together we have the following.

Theorem 3. *Let A be any DGA and let $\mathcal{S}$ be the class of finitely generated semi-free/semi-projective DG A-modules (see 5.3). The abstract notions of semi-projectivity, acyclicity, and semi-flatness from Definition 6 agree with the corresponding DG notions from Definition 8. In the category of DG A-modules, the cotorsion pair generated by $\mathcal{S}$ is complete and it is given by*

$$(\textit{DG-projective DG-modules}, \textit{exact DG-modules})\,.$$

The direct limit closure of $\mathcal{S}$ is the class of semi-flat (or DG-flat) DG-modules.

Proof. Let $\mathcal{P}$ be the class of DG-projective DG-modules, and $\mathcal{E}$ the class of exact DG-modules (i.e. with trivial homology). From Lemma 10 (and by using shift Σ) we have $\mathcal{S}^\perp \subseteq \mathcal{E}$, and from Lemma 9 we have $\mathcal{P} = {}^\perp\mathcal{E}$. Now since $\mathcal{S} \subseteq \mathcal{P}$ we have $\mathcal{E} \subseteq ({}^\perp\mathcal{E})^\perp = \mathcal{P}^\perp \subseteq \mathcal{S}^\perp$, and hence $({}^\perp(\mathcal{S}^\perp), \mathcal{S}^\perp) = (\mathcal{P}, \mathcal{E})$. This shows that the abstract notions of semi-projectivity and acyclicity agree with the corresponding DG notions. Completeness of the cotorsion pair $(\mathcal{P}, \mathcal{E})$ follows from Proposition 2, as already mentioned in Definition 6. It remains to see that the abstract notion of semi-flatness agrees with the corresponding DG notion. It must be shown that if M is a left DG A-module that satisfies $E \otimes_A M \in 1^\perp$, i.e. $\operatorname{Ext}^1_{\mathrm{Ch(Ab)}}(\mathbb{Z}, E \otimes_A M) = 0$, for all acyclic right DG A-modules E, then $E \otimes_A M$ has trivial homology for all such E's. However, by Lemma 10 we have $\operatorname{Ext}^1_{\mathrm{Ch(Ab)}}(\mathbb{Z}, E \otimes_A M) = H_{-1}(E \otimes_A M)$, so the conclusion follows as $- \otimes_A M$ preserves shifts. The last statement in the theorem follows from Theorem 2(2); cf. the discussion in 5.3. □

Remark 7. The cotorsion pair is well-known. It is one of the cotorsion pairs corresponding (via Hovey [18, Thm 2.2]) to the standard projetive model structure on A-DGMod (see for instance Keller [20, Thm 3.2]). That every $S \in \varinjlim \mathcal{S}$ is semi-flat follows directly from results in [2], where it is proved that any semi-projective is semi-flat and that the semi-flats are closed under direct limits. That every semi-flat can be realized as a direct limit of finitely genereated semi-free/projectives is, to the best of my knowledge, new.

5.4. **Ch**(A)**.** In the case of complexes over a ring A a direct calculation using the dual basis theorem component-wise, shows that the dualizable objects in $\mathrm{Ch}(A)$ are precisely the perfect complexes. From above we thus have:

Corollary 5. *Let A be any ring and let $\mathcal{S}$ be the class of perfect A-complexes. In the category* $\mathrm{Ch}(A)$*, the cotorsion pair generated by $\mathcal{S}$ is complete and it is given by (semi-projective complexes, acyclic complexes). The direct limit closure of $\mathcal{S}$ is the class of semi-flat complexes.* □

Remark 8. This cotorsion pair has already been studied for instance in [9] where 2.3.5 and 2.3.6 proves it is a cotorsion pair, and 2.3.25 that it is complete (with slightly different notation). It is not mentioned, however, that it is generated by a set. As already mentioned in the Introduction, the direct limit closure has in this case been worked out in [5].

5.5. **QCoh(X).** Let X be any scheme and let $\mathrm{QCoh}(X)$ be the category of quasi-coherent sheaves (of $\mathscr{O}_X$-modules) on X. This is an abelian and a symmetric monoidal subcategory of $\mathrm{Mod}(X)$ (the category of all sheaves on X), see [15, II Prop. 5.7] and [30, Tag 01CE]. It is also a Grothendieck category, indeed, most of the relevant properties of $\mathrm{QCoh}(X)$ go back to Grothendieck [12, 13]; the existence of a generator is an unpublished result by Gabber (1999), see [30, Tag 077K] and Enochs and Estrada [7] for a proof. The symmetric monoidal category $\mathrm{QCoh}(X)$ is also closed: as explained in [1, 3.7], the internal hom in $\mathrm{QCoh}(X)$ is constructed from that in $\mathrm{Mod}(X)$ composed with the quasi-coherator (the right adjoint of the inclusion $\mathrm{QCoh}(X) \to \mathrm{Mod}(X)$), which always exists [30, Tag 077P].

The *dualizable* objects in QCoh(X) are also studied in Brandenburg [3, Def. 4.7.1 and Rem. 4.7.2], and [3, Prop. 4.7.5] shows that they are exactly the locally free sheaves of finite rank. Recall from Schäppi [29, Def. 6.1.1] (see also [3, Def. 2.2.7]) that a scheme X is said to have the *strong resolution property* if QCoh(X) s generated by locally free sheaves of finite rank. This is the case if X is e.g. a separated noetherian scheme with a family of ample line bundles; see Hovey [17, Prop. 2.3] and Krause [21, Exa. 4.8].

An object $M \in \mathrm{QCoh}(X)$ is *semi-flat* if it is so in the sense of Definition 6, that is, if the functor $- \otimes_{\mathscr{O}_X} M$ is exact and $\mathrm{Ext}^1_{\mathrm{QCoh}(X)}(\mathscr{O}_X, N \otimes_{\mathscr{O}_X} M) = 0$ holds for all $N \in \mathrm{QCoh}(X)$ for which $\mathrm{Ext}^1_{\mathrm{QCoh}(X)}(S, N) = 0$ for all locally free sheaves S of finite rank. Now, from Corollary 2 we get:

Proposition 5. *Let $(X, \mathscr{O}_X)$ be a scheme with the strong resolution property.*

(1) *If $\mathscr{O}_X$ is FP_1, then every semi-flat object in* QCoh(X) *is a direct limit of locally free sheaves of finite rank.*
(2) *If $\mathscr{O}_X$ is FP_2 then, conversely, every direct limit in* QCoh(X) *of locally free sheaves of finite rank is semi-flat.* □

Remark 9. It follows from [14, II Thm. 7.18] that if X is locally noetherian, then every injective object in QCoh(X) is also injective in Mod(X). Thus, in this case one has $\mathrm{Ext}^i_{\mathrm{QCoh}(X)}(M, N) \cong \mathrm{Ext}^i_{\mathrm{Mod}(X)}(M, N)$ for all $M, N \in \mathrm{QCoh}(X)$.

Theorem 4. *Let X be a noetherian scheme with the strong resolution property. In the category* QCoh(X)*, the direct limit closure of the locally free sheaves of finite rank is precisely the class of semi-flat sheaves.*

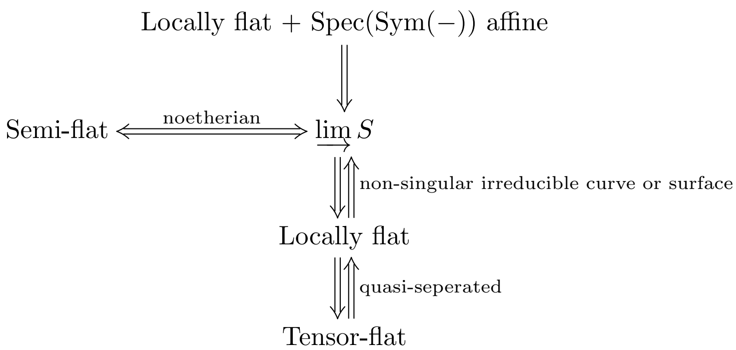
Proof. As X is, in particular, a locally noetherian scheme, Remark 9 and [15, III Prop. 6.3(c)] shows that $\mathrm{Ext}^i_{\mathrm{QCoh}(X)}(\mathscr{O}_X, -) \cong H^i(X, -)$ for all $i \geqslant 0$. If we view $H^i(X, -)$ as a functor $\mathrm{Mod}(X) \to \mathrm{Ab}$, then it preserves direct limits by [15, III Prop. 2.9] as X is a noetherian scheme (see also [15, III 3.1.1]). But then $H^i(X, -)$ also preserves direct limits as a functor $\mathrm{QCoh}(X) \to \mathrm{Ab}$ since colimits in QCoh(X) are just computed in Mod(X), see [30, Tag 01LA]. We conclude that $\mathscr{O}_X$ is both FP_1 and FP_2 and the desired conclusion follows from Proposition 5. □

This is not the first Lazard-like theorem for quasi-coherent sheaves. The usual notion of flatness is *locally flat*, which means that the stalks are flat. Such sheaves are tensor-flat, and the converse holds if the scheme is quasi-separated [3, Lem. 4.6.2].

In [6, (5.4)] Crawley-Boevey proves that $\varinjlim \mathcal{S}$ is precisely the locally flat sheaves if X is a non-singular irreducible curve or surface over a field k.

In [3, 2.2.4] Brandenburg proves that if X has the strong resolution property and M is locally flat and Spec(Sym(M)) is affine, then $M \in \varinjlim \mathcal{S}$.

Thus for a scheme with the strong resolution property we have the relations:



It would be interesting to get a concrete description of the semi-flat, the acyclic and the semi-projective objects in QCoh(X).

Some work has been done in this direction. In Enochs, Estrada, and García-Rozas [8, 3.1] we see that the semi-projective objects are locally projective, and in [8, 4.2] we see that they are precisely the locally projective sheaves in the special case of $P_1(k)$ (the projective line over an algebraically closed field k). In this case a concrete computational description of the (abstract) acyclic objects are given. I am not aware of anybody explicitly studying semi-flat sheaves.

5.6. **Additive functors.** Following [25], let $\mathscr{C}_0 = \mathrm{Ab}$, let $\mathcal{X}$ be a small preadditive category, let $\mathcal{X}^{\mathrm{op}}$ be the dual category, let $\mathscr{C}_L = [\mathcal{X}, \mathrm{Ab}]$ and $\mathscr{C}_R = [\mathcal{X}^{\mathrm{op}}, \mathrm{Ab}]$ be the categories of additive functors, and let $\mathcal{S}$ be the class of finite direct sums of representable functors (recall that the *representable* functors in $\mathscr{C}_L$ and $\mathscr{C}_R$ are the functors $\mathcal{X}(x, -)$ and $\mathcal{X}(-, x)$ where $x \in \mathcal{X}$). We define $\mathcal{X}(-, x)^* = \mathcal{X}(x, -)$ and vice versa. As in [25] one can define a tensorproduct

$$\otimes_{\mathcal{X}} \colon [\mathcal{X}^{\mathrm{op}}, \mathrm{Ab}] \times [\mathcal{X}, \mathrm{Ab}] \longrightarrow \mathrm{Ab}\,.$$

We claim that these data satisfy Setup 1: The categories $\mathscr{C}_L$ and $\mathscr{C}_R$ are Grothendieck and generated by $\mathcal{S}$; see [25, Lem. 2.4]. Note that $\mathcal{S}$ is small as $\mathcal{X}$ is small. Furthermore, $\mathcal{S}$ is closed under extensions; indeed the objects in $\mathcal{S}$ are projective (in fact, every projective object is a direct summand of an object from $\mathcal{S}$), hence any extension is a direct sum. If $\mathcal{X}$ is additive, then $\mathcal{X}(-, x) \oplus \mathcal{X}(-, y) \cong \mathcal{X}(-, x \oplus y)$, so in this case $\mathcal{S}$ is just the class of representable functors (finite direct sums are not needed). Further, as in [25] the tensor product is such that for any $F \in \mathscr{C}_L$ and $G \in \mathscr{C}_R$ we have

$$\mathcal{X}(-, x) \otimes_{\mathcal{X}} F \cong Fx \quad \text{ and } \quad G \otimes_{\mathcal{X}} \mathcal{X}(x, -) \cong Gx\ ,$$

which by the Yoneda lemma, and the fact that $\mathrm{Ab}(1, -)$ is the identity gives the required isomorphisms from Setup 1. As the functors $\mathcal{X}(-, x) \otimes_{\mathcal{X}} ?$ and $? \otimes_{\mathcal{X}} \mathcal{X}(x, -)$ are nothing but evaluation at x, they are exact. Finally, as $1 = \mathbb{Z} \in \mathrm{Ab}$ is projective, Corollary 1 gives a new proof of [25, Thm 3.2]:

Corollary 6. *Let $\mathcal{X}$ be an additive category, and let $\mathcal{S}$ be the finitely generated projective functors or the representable functors in* $[\mathcal{X}, \mathrm{Ab}]$ *(or the direct sums of representable functors if $\mathcal{X}$ is only preadditive). A functor F is flat iff $F \in \varinjlim \mathcal{S}$.*

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Paper II

Direct limit closure of induced quiver representations

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DIRECT LIMIT CLOSURE OF INDUCED QUIVER REPRESENTATIONS

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Abstract. In 2004 and 2005 Enochs et al. characterized the flat and projective quiver-representations of left rooted quivers. The proofs can be understood as filtering the classes $\Phi(\operatorname{Add}\mathscr{X})$ and $\Phi(\varinjlim \mathscr{X})$ when $\mathscr{X}$ is the finitely generated projective modules over a ring. In this paper we generalize the above and show that $\Phi(\mathscr{X})$ can always be filtered for any class $\mathscr{X}$ in any AB5-abelian category. With an emphasis on $\Phi(\varinjlim \mathscr{X})$ we investigate the Gorenstein homological situation. Using an abstract version of Pontryagin duals in abelian categories we give a more general characterization of the flat representations and end up by describing the Gorenstein flat quiver representations over right coherent rings.

Introduction

Let Q be a quiver (i.e. a directed graph) and consider for a class $\mathscr{X}$ of objects in an abelian category $\mathscr{A}$ the class $\Phi(\mathscr{X}) \subseteq \operatorname{Rep}(Q,\mathscr{A})$ of quiver representations. This is the class containing all representations, F, s.t. the canonical map $\bigoplus_{w\to v} F(w) \to F(v)$ is monic and has cokernel in $\mathscr{X}$ for all verteces v - the sum being over all arrows to v. When Q is left-rooted (i.e Q has no infinite sequence of composable arrows of the form $\cdots \to \bullet \to \bullet \to \bullet$) it was observed by Enochs, Oyonarte and Torrecillas in [10] and Enochs and Estrada in [7] that when $\mathscr{A}$ is the category of modules over a ring,

$$\Phi(\operatorname{Proj}(\mathscr{A})) = \operatorname{Proj}(\operatorname{Rep}(Q,\mathscr{A}), \text{ and} \tag{1}$$

$$\Phi(\operatorname{Flat}(\mathscr{A})) = \operatorname{Flat}(\operatorname{Rep}(Q,\mathscr{A}). \tag{2}$$

Here the flat objects are precisely the direct limit closure of the finitely generated projective objects. This was done by showing, that if $\mathscr{X}$ is the finitely generated projective modules over a ring we can filter the classes $\Phi(\operatorname{Add}\mathscr{X})$ and $\Phi(\varinjlim \mathscr{X})$ by sums of objects of the form $f_*(\mathscr{X})$ where $f_v\colon \mathscr{A} \to \operatorname{Rep}(Q,\mathscr{A})$ is the left-adjoint of the evaluation functor $e_v\colon \operatorname{Rep}(Q,\mathscr{A}) \to \mathscr{A}$ at the vertex v. They show

$$\Phi(\operatorname{Add}\mathscr{X}) = \operatorname{Add} f_*(\mathscr{X}) \tag{3}$$

$$\Phi(\varinjlim \mathscr{X}) = \varinjlim \operatorname{add} f_*(\mathscr{X}). \tag{4}$$

In 2014 Holm and Jørgensen [14] generalized (1) to abelian categories with enough projective objects, and combining [14, Thm. 7.4a and 7.9a] with Šťovíček [20, Prop. 1.7] we get the following generalization of (3). If $\mathscr{X}$ is a generating set of objects in a Grothendieck abelian category, then

$$\Phi(\operatorname{sFilt}\mathscr{X}) = \operatorname{sFilt} f_*(\mathscr{X}), \tag{5}$$

where $\operatorname{sFilt}\mathscr{X}$ consists of all summands of $\mathscr{X}$-filtered objects. In this paper we show that $\Phi(\mathscr{X})$ can always be filtered by $f_*(\mathscr{X})$. We have the following:

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Theorem A. *Let $\mathscr{A}$ be an AB5-abelian category, let $\mathscr{X} \subset \mathscr{A}$ and let Q be a left-rooted quiver. Then*

i) Any $F \in \Phi(\mathscr{X})$ is $f_*(\mathscr{X})$-filtered.

If $\mathscr{X}$ is closed under filtrations, then

ii) $\Phi(\mathscr{X}) = \operatorname{Filt} f_*(\mathscr{X})$

In particular we have the following.

iii) $\Phi(\operatorname{Filt} \mathscr{X}) = \operatorname{Filt} f_*(\mathscr{X}) = \operatorname{Filt} \Phi(\mathscr{X})$
iv) $\Phi(\operatorname{sFilt} \mathscr{X}) = \operatorname{sFilt} f_*(\mathscr{X}) = \operatorname{sFilt} \Phi(\mathscr{X})$

If $X \subseteq FP_{2.5}(\mathscr{A})$ and $\mathscr{A}$ is locally finitely presented, then

v) $\Phi(\varinjlim \mathscr{X}) = \varinjlim \operatorname{ext} f_*(\mathscr{X}) = \varinjlim \Phi(\mathscr{X})$

Here $FP_{2.5}(\mathscr{A})$ is a certain class of objects which sits between $FP_2(\mathscr{A})$ and $FP_3(\mathscr{A})$ with the property that it is always closed under extensions. In many situations (e.g $\mathscr{A}$ = R-Mod) $FP_{2.5}(\mathscr{A}) = FP_2(\mathscr{A})$ (Lemma 1.4).

We note that $\varinjlim \operatorname{ext} \mathscr{X} = \varinjlim \operatorname{add} \mathscr{X}$ and $\operatorname{Add} \mathscr{X} = \operatorname{sFilt} \mathscr{X}$ when $\mathscr{X}$ consists of projective objects and that the finitely generated projective objects are FP_n for any n. Theorem A is thus a generalization of (3) and (4). It also generalizes (5) to arbitrary classes in not neccesarily Gorenstein abelian categories. We show how to use this to reprove (1) in abelian categories with enough projective objects. We also show (2) (Lemma 2.12) when the category is generated by finitely generated projective objects and flat is understood as their direct limit closure (see Theorem C however for a more general version).

We then apply Theorem A v) to the Gorenstein homological situation. We let $\operatorname{GProj}(\mathscr{A})$ be the Gorenstein projective objects, let $\operatorname{Gproj}(\mathscr{A}) = \operatorname{GProj}(\mathscr{A}) \cap FP_{2.5}(\mathscr{A})$ and immediately get $\Phi(\varinjlim \operatorname{Gproj}(\mathscr{A})) = \varinjlim \operatorname{ext} f_*(\operatorname{Gproj}(\mathscr{A}))$. Contrary to the case for ordinary projective objects, it is not clear, that this equals $\varinjlim \operatorname{Gproj}(\operatorname{Rep}(Q, \mathscr{A}))$ without some restrictions on Q. In the following target-finite means that there are only finitely many arrows with a given target and locally path-finite means that there are only finitely many paths between two given vertices. We have

Theorem B. *Let $\mathscr{A}$ be a locally finitely presented category with enough projective objects, let Q be a left-rooted quiver and assume that either*

- *Q is target-finite and locally path-finite, or*
- $\varinjlim \operatorname{Gproj}(\mathscr{A}) = \varinjlim \operatorname{GProj}(\mathscr{A})$ *(e.g if $\mathscr{A}$ =* R-Mod *and R is Iwanaga-Gorenstein).*

Then

$$\Phi(\varinjlim \operatorname{Gproj}(\mathscr{A})) = \varinjlim \operatorname{Gproj}(\operatorname{Rep}(Q, \mathscr{A})) = \varinjlim \Phi(\operatorname{Gproj}(\mathscr{A})).$$

In the latter case, this equals $\varinjlim \operatorname{GProj}(\operatorname{Rep}(Q, \mathscr{A}))$.

Again contrary to the ordinary projective objects even for $\mathscr{A}$ = R-Mod it is not true in general that $\varinjlim \operatorname{Gproj}(\mathscr{A})$ is all the Gorenstein Flat objects, $\operatorname{GFlat}(\mathscr{A})$, nor those objects with Gorenstein injective Pontryagin dual, $\operatorname{wGFlat}(\mathscr{A})$. In the rest of the paper we study these classes in $\operatorname{Rep}(Q, \mathscr{A})$. First we must explain what we mean by an abstract Pontryagin dual and we show how these arise natually and agree with the standard notion in well-known abelian categories. We go on and characterize those objects with injective (or Gorenstein injective) Pontryagin dual as follows.

Theorem C. *Let $\mathscr{A}$ be an abelian category with a Pontryagin dual to a category with enough injective objects and let Q be a left-rooted quiver. Then*

$$\operatorname{Flat}(\operatorname{Rep}(Q,\mathscr{A})) = \Phi(\operatorname{Flat}(\mathscr{A}))$$
$$\operatorname{wGFlat}(\operatorname{Rep}(Q,\mathscr{A})) = \Phi(\operatorname{wGFlat}(\mathscr{A}))$$

Here $\operatorname{Flat}(\mathscr{A})$ is those objects with injective Pontryagin dual so this result reproves (2) using the simpler characterization of injective representations in Enochs, Estrada and García Rozas [8, Prop 2.1] instead of going through the proof of (4) as in [10]. Theorem C tells us that, under the conditions of Theorem B, if $\varinjlim \operatorname{Gproj}(\mathscr{A}) = \operatorname{wGFlat}(\mathscr{A})$ then also $\varinjlim \operatorname{Gproj}(\operatorname{Rep}(Q,\mathscr{A})) = \operatorname{wGFlat}(\operatorname{Rep}(Q,\mathscr{A}))$. (Corollary 4.7)

In [8] it is proved that $\operatorname{wGFlat}(\operatorname{Rep}(Q,\mathscr{A})) = \operatorname{GFlat}(\operatorname{Rep}(Q,\mathscr{A}))$ when $\mathscr{A} =$ R-Mod and R is Gorenstein. We end this paper by showing that this also hold if R is just assumed to be coherent if we impose proper finiteness conditions on Q.

Theorem D. *Let R be a right coherent ring and let Q be a left-rooted and target-finite quiver. Then*

$$\operatorname{wGFlat}(\operatorname{Rep}(Q,\text{R-Mod})) = \operatorname{GFlat}(\operatorname{Rep}(Q,\text{R-Mod})).$$

See also Proposition 5.6 for a version for abelian categories. If Q is further locally path-finite (or R is Gorenstein and Q is just assumed to be left-rooted) the conditions for Theorem B and Theorem C are satisfied as well, so in this case (Corollary 5.8) if $\varinjlim \operatorname{Gproj}(\mathscr{A}) = \operatorname{GFlat}(\mathscr{A})$ then

$$\varinjlim \operatorname{Gproj}(\operatorname{Rep}(Q,\mathscr{A})) = \operatorname{GFlat}(\operatorname{Rep}(Q,\mathscr{A})) = \Phi(\operatorname{GFlat}(\mathscr{A})).$$

The equality $\varinjlim \operatorname{Gproj}(\text{R-Mod}) = \operatorname{GFlat}(\text{R-Mod})$ is known to hold when R is an Iwanaga-Gorenstein ring (Enochs and Jenda [9, Thm. 10.3.8]) or if R is an Artin algebra which is virtually Gorenstein (Beligiannis and Krause [3, Thm. 5]). In general $\varinjlim \operatorname{Gproj}(\text{R-Mod})$ and $\operatorname{GFlat}(\text{R-Mod})$ are different (Holm and Jørgensen [13, Thm. A]).

1. Locally finitely presented categories

In the following let $\mathscr{A}$ be an abelian category. First we recall some basic notions.

We say $\mathscr{A}$ is (AB4) if $\mathscr{A}$ is cocomplete and forming coproducts is exact, (AB4*) if $\mathscr{A}$ is complete and forming products is exact, (AB5) if filtered colimits are exact, Grothendieck if it is (AB5) and has a generator (i.e. a generating object or equivalently a generating set). Here a class $\mathscr{S} \subseteq \mathscr{A}$ is said to generate $\mathscr{A}$ if it detects zero-morphisms i.e. a morphism $X \xrightarrow{f} Y$ is zero iff $S \xrightarrow{g} X \xrightarrow{f} Y$ is zero for all g with $S \in \mathscr{S}$.

We write $X \in \varinjlim \mathscr{X}$ if $X = \varinjlim X_i$ for some filtered system $\{X_i\} \subseteq \mathscr{X}$. We write $X \in \operatorname{Filt} \mathscr{X}$ if there is a chain $X_0 \subseteq \ldots \subseteq X_\lambda = X$ for some ordinal λ s.t. $X_{\alpha+1}/X_\alpha \in \mathscr{X}$ for all $\alpha < \lambda$ and $\varinjlim_{\alpha<\alpha_0} X_\alpha = X_{\alpha_0}$, for any limit ordinals $\alpha_0 \leq \lambda$. We say $X \in \operatorname{Filt} \mathscr{X}$ is *$\mathscr{X}$-filtered.* When λ is finite, we say X is a *finite extension* of (objects of) $\mathscr{X}$, and we let $\operatorname{ext}(\mathscr{X})$ denote the class of finite extensions of $\mathscr{X}$. This is also the extension closure of $\mathscr{X}$ i.e. the smallest subcategory of $\mathscr{A}$ containing $\mathscr{X}$ and closed under extensions. For example the class $\bigoplus \mathscr{X}$ is the class of all (infinite) sums of elements of $\mathscr{X}$. Such a sum, $\bigoplus_{i=1}^{\lambda} X_i$ is a colimit of a diagram with no arrows, and as such is neither a direct limit nor a filtration. It can however be realized as a filtration by $\{\bigoplus_{i=1}^{\alpha} X_i\}$, for $\alpha < \lambda$ and as a direct limit as $\{\bigoplus_{i\in I} X_i\}$, for I finite, with arrows the inclusions. In fact $\bigoplus \mathscr{X} = \operatorname{Filt} \mathscr{X}$ when $\mathscr{X}$ consists of projective objects. We say that $X \in \mathscr{A}$ is FP_n if the canonical map

$$\varinjlim \operatorname{Ext}^k(X, Y_i) \to \operatorname{Ext}^k(X, \varinjlim Y_i)$$

is an isomorphism for every $0 \leq k < n$. The objects $FP_1(\mathscr{A})$ are called finitely presented, and the objects s.t the above map is injective for $k = 0$ is called finitely generated and denoted $FP_0(\mathscr{A})$. The category $\mathscr{A}$ is called locally finitely presented if it satisfies one (and therefore all) of the following equivalent conditions:

(i) $FP_1(\mathscr{A})$ is skeletally small (i.e. the isomorphism classes form a set) and $\varinjlim FP_1(\mathscr{A}) = \mathscr{A}$ (Crawley-Boevey [6])
(ii) $\mathscr{A}$ is Grothendieck and $FP_1(\mathscr{A})$ generate $\mathscr{A}$. (Breitsprecher [4])
(iii) $\mathscr{A}$ is Grothendieck and $\varinjlim FP_1(\mathscr{A}) = \mathscr{A}$ ([4]).

The direct limit is very well-behaved in locally finitely presented categories. In particular we have that if $\mathscr{X} \subseteq FP_1(\mathscr{A})$ is closed under direct sums, then $\varinjlim \mathscr{X}$ is closed under direct limits, and is thus the direct limit closure of $\mathscr{X}$ [6, Lemma p. 1664]. We also have the following. The proof was communicated to me by Jan Šťovíček (any mistakes are mine).

Proposition 1.1. *Let $\mathscr{A}$ be a locally finitely presented abelian category. If $\mathscr{X} \subseteq FP_2(\mathscr{A})$ is closed under extensions then so is $\varinjlim \mathscr{X}$. It is thus closed under filtrations.*

Proof. Let $\{S_i\}, \{T_j\} \subseteq \mathscr{X}$ be directed systems and let

$$0 \to \varinjlim S_i \to E \to \varinjlim T_j \to 0$$

be an exact sequence. We want to show that $E \in \varinjlim \mathscr{X}$. First by forming the pullback

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varinjlim S_i & \longrightarrow & E_j & \longrightarrow & T_j & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & \ulcorner & \downarrow & & \\ 0 & \longrightarrow & \varinjlim S_i & \longrightarrow & E & \longrightarrow & \varinjlim T_j & \longrightarrow & 0 \end{array}$$

we see that $E = \varinjlim E_j$ since $\mathscr{A}$ is AB5 as it is locally finitely presented abelian, hence Grothendieck. Now since T_j is in $FP_2(\mathscr{A})$ for every j we have that

$$[0 \to \varinjlim S_i \to E_j \to T_j \to 0] \in \operatorname{Ext}^1(T_j, \varinjlim S_i)$$

is in the image of the canonical map from $\varinjlim \operatorname{Ext}^1(T_j, E_i)$, that is, it is a pushout

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_i & \longrightarrow & E_{ij} & \longrightarrow & T_j & \longrightarrow & 0 \\ & & \downarrow & \lrcorner & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \varinjlim S_i & \longrightarrow & E_j & \longrightarrow & T_j & \longrightarrow & 0 \end{array}$$

for some i and some extension $E_{ij} \in \mathscr{A}$.

Now construct for every $k \geq i$ the pushout

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_i & \longrightarrow & E_{ij} & \longrightarrow & T_j & \longrightarrow & 0 \\ & & \downarrow & \lrcorner & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S_k & \longrightarrow & E_{kj} & \longrightarrow & T_j & \longrightarrow & 0 \end{array}$$

Then $\varinjlim_k E_{kj} = E_j$ so $E_j \in \varinjlim \mathscr{X}$ as $E_{kj} \in \mathscr{X}$ when $\mathscr{X}$ is closed under extensions.

Finally $E = \varinjlim E_j \in \varinjlim \mathscr{X}$ as $\varinjlim \mathscr{X}$ is closed under direct limits when $\mathscr{X} \subset FP_1(\mathscr{A})$. □

The classes $FP_n(\mathscr{A})$ are all closed under finite sums (as in [4, Lem. 1.3]). They are not necessarily closed under extensions, but the following subclasses are:

Definition 1.2. Let $\mathscr{A}$ be an abelian category. We say $X \in \mathscr{A}$ is $FP_{n.5}$ if X is FP_n and furthermore, that the natural map $\varinjlim \operatorname{Ext}^n(X, Y_i) \to \operatorname{Ext}^n(X, \varinjlim Y_i)$ is monic for every filtered system $\{Y_i\} \subseteq \mathscr{A}$. We let FP_* stand for an unspecified (but fixed) FP_n or $FP_{n.5}$

Note that by definition $FP_0(\mathscr{A}) = FP_{0.5}(\mathscr{A})$ and also $FP_1(\mathscr{A}) = FP_{1.5}(\mathscr{A})$ by Stenström [19, Prop. 2.1] when $\mathscr{A}$ is AB5. We have the following generalization of [4, Lem. 1.9] for $n, * = 1$ and $\mathscr{A}$ Grothendieck.

Lemma 1.3. *Let $\mathscr{A}$ be an AB5-abelian category and let*

$$0 \to A \to B \to C \to 0$$

be an exact sequence. Then

(i) If A and C are $FP_{n.5}$, then so is B.
(ii) If B is FP_* then A is FP_{*-1} iff C is FP_*.

Proof. (i) Let $\{X_i\} \subset \mathscr{A}$ be a filtered system. From the long exact sequence in homology we get for all $k < n$:

$$\begin{array}{ccccccccc}
\varinjlim \operatorname{Ext}^{k-1}(A, X_i) & \to & \varinjlim \operatorname{Ext}^{k}(C, X_i) & \to & \varinjlim \operatorname{Ext}^{k}(B, X_i) & \to & \varinjlim \operatorname{Ext}^{k}(A, X_i) & \to & \varinjlim \operatorname{Ext}^{k+1}(C, X_i) \\
\downarrow \cong & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \downarrow \\
\operatorname{Ext}^{k-1}(A, \varinjlim X_i) & \to & \operatorname{Ext}^{k}(C, \varinjlim X_i) & \to & \operatorname{Ext}^{k}(B, \varinjlim X_i) & \to & \operatorname{Ext}^{k}(A, \varinjlim X_i) & \to & \operatorname{Ext}^{k+1}(C, \varinjlim X_i)
\end{array}$$

and

$$\begin{array}{ccccccc}
\varinjlim \operatorname{Ext}^{n-1}(A, X_i) & \to & \varinjlim \operatorname{Ext}^{n}(C, X_i) & \to & \varinjlim \operatorname{Ext}^{n}(B, X_i) & \to & \varinjlim \operatorname{Ext}^{n}(A, X_i) \\
\downarrow \cong & & \downarrow & & \downarrow & & \downarrow \\
\operatorname{Ext}^{n-1}(A, \varinjlim X_i) & \to & \operatorname{Ext}^{n}(C, \varinjlim X_i) & \to & \operatorname{Ext}^{n}(B, \varinjlim X_i) & \to & \operatorname{Ext}^{n}(A, \varinjlim X_i)
\end{array}$$

And the result follows by the 5-lemma. (ii) is proved similarly. Note that when $* = 1$ we must use that $FP_1 = FP_{1.5}$ because $FP_0 = FP_{0.5}$. □

Lemma 1.4. *Let $\mathscr{A}$ be an AB5-abelian category generated by a set of $FP_{n.5}$-objects. Then*

(i) If $X \in FP_0(\mathscr{A})$ there exists an epi $X_0 \to X$ with $X_0 \in FP_{n.5}(\mathscr{A})$.
(ii) $FP_k(\mathscr{A}) = FP_{k.5}(\mathscr{A})$ for all $k \leq n$

Proof. For (i) notice that by [4, satz 1.6] if $\mathscr{A}$ is generated by $\mathscr{X} \subseteq FP_1(\mathscr{A})$ and $C \in FP_0(\mathscr{A})$ then we have an epi from a finite sum of elements of $\mathscr{X}$ to C. But FP_n (and $FP_{n.5}$) are all closed under finite sums. The proof of (ii) goes by induction. The case $n = 0$ is true by definition, so assume $\mathscr{A}$ is generated by a set of $FP_{n.5}$-objects and that $X \in FP_n(\mathscr{A})$. By (i) we get an exact sequence

$$0 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow X \longrightarrow 0$$

with $X_0 \in FP_{n.5}(\mathscr{A})$. By Lemma 1.3 (ii) $X_1 \in FP_{n-1}(\mathscr{A})$ which by induction hypothesis equals $FP_{(n-1).5}(\mathscr{A})$ so $X \in FP_{n.5}(\mathscr{A})$ again by Lemma 1.3 (ii). □

In particular $FP_{n.5}(\text{R-Mod}) = FP_n(\text{R-Mod})$ is closed under extensions for any n and any ring R. We think of the objects of $FP_*(\mathscr{A})$ as beeing small.

2. Quiver representations

Let Q be a quiver, i.e. a directed graph. We denote the vertices by Q_0 and we denote an arrow (resp. a path) from w to v by $w \to v$ (resp. $w \rightsquigarrow v$). A quiver may have infinitely many vertices and arrows, but we will need the following finiteness conditions.

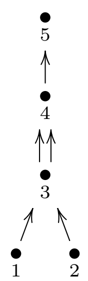
Definition 2.1. Let Q be a quiver. We say Q is *target-finite* (resp. *source-finite*) if there are only finitely many arrows with a given target (resp. source). We say Q is *left-rooted* (resp. *right-rooted*) if there is no infinite sequence of composable arrows $\cdots \to \bullet \to \bullet$ (resp. $\bullet \to \bullet \to \cdots$). Finally we say Q is *locally path-finite* if there is only finitely many paths between any two given vertices.

Remark 2.2. Notice that Q is target-finite (resp. left-rooted) iff Q^{op} is source-finite (resp. right-rooted) and that left/right-rooted quivers are necessarily acyclic (i.e have no cycles or loops). Locally path-finite is self-dual. Even if a quiver satisfies all of the above finiteness conditions, it can still have infinitely many vertices and arrows, e.g the quiver $\cdots \leftarrow \bullet \to \bullet \leftarrow \bullet \to \bullet \leftarrow \bullet \to \cdots$

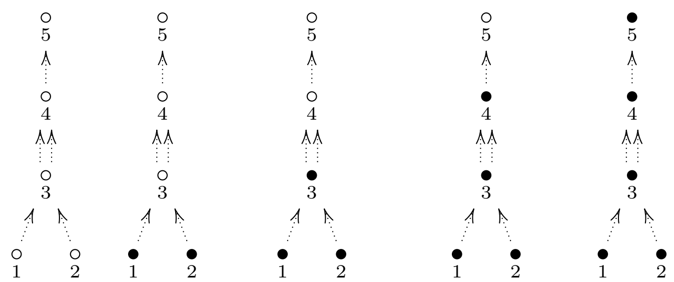
When the quiver is left-rooted we can use the following sets for inductive arguments. Let $V_0 = \emptyset$ and define for any ordinal λ, $V_{\lambda+1} = \{v \in Q_0 | w \to v \Rightarrow w \in V_\lambda\}$ and for limit ordinals $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$. Notice that V_1 is precisely the sources of Q.

As noted in [10, Prop. 3.6] a quiver is left-rooted precisely when $Q_0 = V_\lambda$ for some λ.

Example 2.3. Let Q be the (left-rooted) quiver:



For this quiver, the transfinite sequence $\{V_\alpha\}$ looks like this:



$V_0 = \varnothing$ $V_1 = \{1,2\}$ $V_2 = \{1,2,3\}$ $V_3 = \{1,2,3,4\}$ $V_4 = Q_0$

Let now $\mathscr{A}$ be an abelian category. A quiver Q generates a category $\overline{Q}$, called the path category, with objects Q_0 and morphisms the paths in Q. We define $\mathrm{Rep}(Q, \mathscr{A}) = \mathrm{Fun}(\overline{Q}, \mathscr{A})$. Note that $F \in \mathrm{Rep}(Q, A)$ is given by its values on vertices and arrows and we picture F as a Q-shaped diagram in $\mathscr{A}$.

For $v \in Q_0$ the evaluation functor $e_v : \mathrm{Rep}(Q, \mathscr{A}) \to \mathscr{A}$ is given by $e_v(F) = F(v)$ for $v \in Q_0$ and $e_v(\eta) = \eta_v$ for $\eta\colon F \to G$. If $\mathscr{A}$ has coproducts (or Q is locally path-finite) this has a left-adjoint $f_v : \mathscr{A} \to \mathrm{Rep}(Q, \mathscr{A})$ given by

$$f_v(X)(w) = \bigoplus_{v \rightsquigarrow w} X$$

where the sum is over all paths from v to w and $f_v(X)(w \to w')$ is the natural inclusion. For $\mathscr{X} \subseteq \mathscr{A}$ we define

$$f_*(\mathscr{X}) = \{f_v(X) \mid v \in Q_0, X \in \mathscr{X}\} \subseteq \mathrm{Rep}(Q, \mathscr{A}).$$

See [10] or [14] for details.

Remark 2.4. Limits and colimits are point-wise in $\mathrm{Rep}(Q, \mathscr{A})$, so e_v preserves them and is in particular exact. Thus its left-adjoint f_v preserves projective objects.

Definition 2.5. For any quiver Q, any abelian category $\mathscr{A}$, any $F \in \mathrm{Rep}(Q, \mathscr{A})$ and any $v \in Q_0$ we have a canonical map $\varphi_v^F = \bigoplus_{w \to v} F(w) \to F(v)$ and we set

$$\Phi(\mathscr{X}) = \left\{F \in \mathrm{Rep}(Q, \mathscr{A}) \middle| \forall v \in Q_0 : \varphi_v^F \text{ is monic and } \operatorname{coker} \varphi_v^F \in \mathscr{X}\right\}.$$

Remark 2.6. Observe that $f_v(\mathscr{X}) \subseteq \Phi(\mathscr{X})$. In fact for any $v \in Q_0$, $\varphi_w^{f_v(X)}$ is an isomorphism, unless $w = v$ in which case it is monic (in fact zero if Q is acyclic) with cokernel X. As in [14, Prop. 7.3] if Q is left-rooted then $\Phi(\mathscr{X}) \subseteq \mathrm{Rep}(Q, \mathscr{X})$ if $\mathscr{X}$ is closed under arbitrary sums or Q is locally path-finite and $\mathscr{X}$ is closed under finite sums.

The aim of this section is to show that sums of objects of $f_*(\mathscr{X})$ filter $\Phi(\mathscr{X})$. Let us first see how f and Φ play together with various categorical constructions.

Lemma 2.7. *Let Q be a quiver, $\mathscr{A}$ an abelian category satisfying AB4, and $\mathscr{X} \subseteq \mathscr{A}$ arbitrary. Then*

(i) $f_*(\text{extensions of } \mathscr{X}) \subseteq \text{extensions of } f_*(\mathscr{X})$,
(ii) $f_*(\text{summands of } \mathscr{X}) \subseteq \text{summands of } f_*(\mathscr{X})$,
(iii) $f_*(\varinjlim \mathscr{X}) \subseteq \varinjlim f_*(\mathscr{X})$,
(iv) $f_*(\operatorname{Filt} \mathscr{X}) \subseteq \operatorname{Filt} f_*(\mathscr{X})$.

Proof. (i) follows since f_v is exact when $\mathscr{A}$ is AB4 and (iii) since f_v is a left adjoint. (ii) is clear and (iv) follows from (i) and (iii). □

Lemma 2.8. *Let again Q be a quiver, $\mathscr{A}$ an abelian category satisfying AB4, and $\mathscr{X} \subseteq \mathscr{A}$ arbitrary. Then*

(i) $\Phi(\text{extensions of } \mathscr{X}) \subseteq \text{extensions of } \Phi(\mathscr{X})$,
(ii) $\text{summands of } \Phi(\mathscr{X}) \subseteq \Phi(\text{summands of } \mathscr{X})$.

When $\mathscr{A}$ is AB5 we further have

(iii) $\varinjlim \Phi(\mathscr{X}) \subseteq \Phi(\varinjlim \mathscr{X})$,
(iv) $\operatorname{Filt} \Phi(\mathscr{X}) \subseteq \Phi(\operatorname{Filt} \mathscr{X})$.

When $\mathscr{A}$ is AB4 and Q is target-finite we have*

(v) $\prod \Phi(\mathscr{X}) \subseteq \Phi(\prod \mathscr{X})$.

Proof. (ii) follows as retracts respects kernels and cokernels, (iii) is clear when $\mathscr{A}$ satisfies AB5. For (i) let $0 \to F \to F'' \to F' \to 0$, be an exact sequence with $F, F' \in \Phi(\mathscr{X})$. For every $v \in Q_0$ we have that

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \oplus_{w\to v}F(w) & \longrightarrow & \oplus_{w\to v}F''(w) & \longrightarrow & \oplus_{w\to v}F'(w) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F(v) & \longrightarrow & F''(v) & \longrightarrow & F'(v) & \longrightarrow & 0\\
 & & \downarrow & & & & \downarrow & & \\
 & & C & & & & C' & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

has exact rows since $\mathscr{A}$ is AB4 and e_v is exact. The condition follows from the snake lemma, since $C, C' \in \mathscr{X}$.

Again (iv) follows from (i) and (iii). For (v) we notice that for any $\{F_i\} \subset \mathscr{A}$ and vertex v we have $\prod_i \phi_v^{F_i} = \phi_v^{\prod F}$ since the sum in the definition of ϕ is finite, hence a product, when Q is target-finite. □

As for smallness we have the following

Lemma 2.9. *Let $\mathscr{A}$ be an abelian category.*

(i) If $\mathscr{A}$ satisfies AB5 then f_v preserves FP_*
(ii) If Q is locally path-finite, then $e_v(-)$ preserves FP_*.
(iii) If Q is target-finite and locally path-finite then

$$\Phi(\mathscr{X}) \cap FP_*(\mathrm{Rep}(Q, \mathscr{A})) \subseteq \Phi(\mathscr{X} \cap FP_*(\mathscr{A})).$$

Proof. (i) This follows from the natural isomorphism ([14, prop 5.2])

$$\mathrm{Ext}^i(f_v(X), -) \cong \mathrm{Ext}^i(X, e_v(-))$$

and the fact that e_v preserves filtered colimits (Remark 2.4).

(ii) In this case e_v has a right adjoint $g_v(X)(w) = \prod_{w \rightsquigarrow v} X$ (see [14, 3.6]) which is a finite product, hence a sum, as Q is locally path-finite. So $g_v(-)$ preserves filtered colimits. Thus e_v preserves FP_*, by the natural isomorphism ([14, prop 5.2])

$$\mathrm{Ext}^1(e_v(X), -) \cong \mathrm{Ext}^1(X, g_v(-))$$

(iii) Let $F \in \Phi(\mathscr{X})$ be FP_*. Given $v \in Q_0$ we only need to show that $\mathrm{coker}\phi_v^F$ is FP_*. Since Q is target-finite, $\oplus_{w\to v}F(w)$ is a finite sum of FP_*-objects by (ii) and since FP_* is closed under finite sums the result follows from (ii) and Lemma 1.3 (ii). □

The following two lemmas will be used to construct a $\oplus f_*(\mathscr{X})$-filtration for any $F \in \Phi(\mathscr{X})$ for suitable $\mathscr{X} \subset \mathscr{A}$ when Q is left-rooted. This is the key in proving Theorem A.

Lemma 2.10. *Let Q be an acyclic (e.g. left-rooted) quiver and $\mathscr{A}$ an abelian category satisfying AB4. If $F \in \Phi(\mathscr{X})$ there exists a subrepresentation $F' \subseteq F$ such that*

(a) $F' \in \bigoplus f_*(\mathscr{X})$,
(b) $F'(v) = F(v) \quad \forall u \in V^F = \{v \in Q_0 | w \to v \Rightarrow F(w) = 0\}$,
(c) $F/F' \in \Phi(\mathscr{X})$, with $\mathrm{coker}\,\phi_v^{F/F'} = \mathrm{coker}\,\phi_v^F$ when $v \notin V^F$.

Proof. Define $F' = \bigoplus_{v \in V^F} f_v(F(v))$. We wish to prove that F' is a subrepresentation and that it satisfies (a)-(c).

Clearly F' satisfy (a). To see (b) it suffices to prove, that for any non-trivial path $w \rightsquigarrow v$ with $v \in V^F$ we have $F(w) = 0$ - because then for any $v \in V^F$ we have $f_v(F(v))(v) = F(v)$ and $f_w(F(w))(v) = 0$, $w \neq v$. So let $v \in V^F$ and assume there is a path $w \overset{p}{\rightsquigarrow} w' \to v$. Then $F(w') = 0$ as $v \in V^F$. But then also $F(w) = 0$ as $F(p)$ is monic since $F \in \Phi(\mathscr{X})$.

To see that F' is a subrepresentation satisfying (c) we use the map $F' \to F$ induced by the counits $f_v e_v(F) \to F$. If v is not reachable from V^F (i.e. there is no path $w \rightsquigarrow v$ with $w \in V^F$) this is trivial since then $F'(v) = 0$.

So let Q' be the subquiver consisting of all vertices Q'_0 reacheable from V^F (i.e. $Q'_0 = \{v \in Q_0 \mid \exists w \rightsquigarrow v, w \in V^F\}$ with arrows $\{w \to v \mid w \in Q'_0\}$). We want for all $V \in Q'_0$ that there are exact sequences

(1) $0 \to F'(v) \to F(v)$

(2) $0 \to \bigoplus_{w \to v} F/F'(w) \to F/F'(v) \to \operatorname{coker} \phi_v^F \to 0$ when $v \notin V^F$.

Since Q is acyclic, Q' is left-rooted with sources V^F. We can thus proceed by induction on the sets V'_λ. The case $v \in V'_1 = V_F$ is taken care of by (b), so assume (1) for all $w \in V'_\alpha$ and all $\alpha < \lambda \neq 1$, and let $v \in V'_\lambda$. Then we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \longrightarrow & \oplus_{w \to v} F'(w) & \longrightarrow & F'(v) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \oplus_{w \to v} F(w) & \longrightarrow & F(v) & \longrightarrow & \operatorname{coker} \phi_v^F & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & \oplus_{w \to v} F/F'(w) & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

The first row is exact as $v \notin V^F$ (see Remark 2.6), the second as $F \in \Phi(\mathscr{X})$ and the first column by induction hypothesis and the assumption that $\mathscr{A}$ is $AB4$. Now (1) and (2) follows for $v \in V_\lambda$ by the snake lemma. □

Lemma 2.11. *Let Q be an acyclic quiver, $\mathscr{A}$ an AB5-abelian category, and let $\mathscr{X} \subseteq \mathscr{A}$. Then for any $F \in \Phi(\mathscr{X})$ there exists a chain $0 = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_\lambda \subseteq \ldots \subseteq F$ of subrepresentations of F, such that for all ordinals λ*

(a) $F_\lambda/F_\alpha \in \bigoplus f_*(\mathscr{X})$, *if* $\lambda = \alpha + 1$

(b) $F_\lambda(v) = F(v)$ for all $v \in \bigcup_{\alpha<\lambda} V^{F/F_\alpha}$

(c) $F/F_\lambda \in \Phi(\mathscr{X})$ *with* $\operatorname{coker} \phi^{F/F_\lambda} = \operatorname{coker} \phi_v^F$ *for* $v \notin \bigcup_{\alpha<\lambda} V^{F/F_\alpha}$.

Notice that $\bigcup_{\alpha<\beta+1} V^{F/F_\alpha} = V^{F/F_\beta}$

Proof. We will construct such a filtration by transfinite induction. $0 = F_0$ is evident so assume F_α satisfying (a)-(c) has been constructed for all $\alpha < \lambda$

If $\lambda = \alpha + 1$ then by Lemma 2.10 we have an $F' \subseteq F/F_\alpha$ s.t. $F' \in \bigoplus f_*(\mathscr{X})$ and s.t. $F'' = (F/F_\alpha)/F' \in \Phi(\mathscr{X})$ satisfies

$$F''(v) = 0 \text{ for all } v \in V^{F/F_\alpha} = \{v \in Q_0 | w \to v \implies F(w) = F_\alpha(w)\}.$$

and

$$\operatorname{coker} \phi_v^{F''} = \operatorname{coker} \phi_v^{F/F_\alpha} = \operatorname{coker} \phi_v^F \text{ for all } v \notin V^{F/F_\alpha}$$

Now let F_λ be the pullback

$$\begin{array}{ccc} F_\lambda & \longrightarrow & F' \\ \downarrow & \ulcorner & \downarrow \\ F & \longrightarrow & F/F_\alpha \end{array}$$

Then (a) follows as $F_\lambda/F_\alpha \cong F'$ and (b) and (c) follows since $F/F_\lambda \cong F''$.

If λ is a limit ordinal, we set

$$F_\lambda = \bigcup_{\alpha<\lambda} F_\alpha$$

so that

$$F(v) = F_\lambda(v) \text{ when } v \in \bigcup_{\alpha<\lambda} V^{F/F_\alpha}$$

Then (a) is void and we get (b) by noting that when $v \in V^{F/F_\alpha}$ for some $\alpha < \lambda$, then $F_\lambda(v)$ is the limit of a filtration eventually equal to $F(v)$

$$F_\lambda(v) = e_v\left(\bigcup_{\alpha<\lambda} F_\alpha\right) = \bigcup_{\alpha<\lambda} F_\alpha(v) = F(v).$$

To prove (c) we similarly notice that $\phi_v^{F/\varinjlim F_\alpha} = \varinjlim \phi_v^{F/F_\alpha}$ is monic for any vertex v as $\mathscr{A}$ is AB5 and when $v \notin \bigcup_{\alpha<\lambda} V^{F/F_\alpha}$ then

$$\operatorname{coker} \phi_v^{F/\varinjlim F_\alpha} = \varinjlim \operatorname{coker} \phi_v^{F/F_\alpha} = \varinjlim \operatorname{coker} \phi_v^F = \operatorname{coker} \phi_v^F$$

□

The following figure shows an example of this construction.

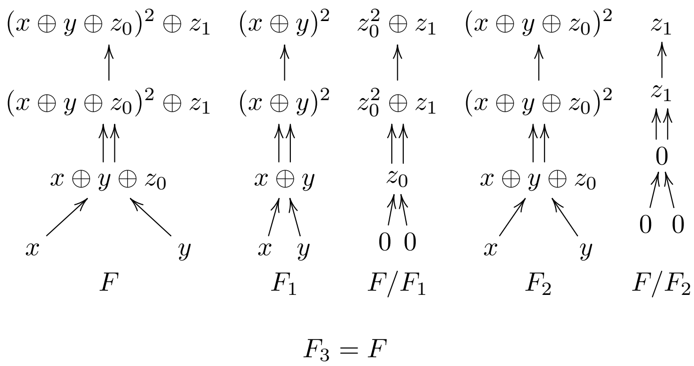


FIGURE 1. Example of the construction of the subrepresentations F_α

We can now proof Theorem A from the introduction.

Proof of Theorem A.

i) Let $F \in \Phi(\mathscr{X})$ and let $\{F_\lambda\}$ be the filtration of Lemma 2.11. First we show that $F_\lambda(v) = F(v)$ for all $v \in V_\lambda$. The case $\lambda = 0$ is trivial, so let $\lambda = \alpha + 1$, assume $F_\alpha(v) = F(v)$. and let $v \in V_\lambda$. Then for paths $w \to v$ we have $w \in V_\alpha$ so $F_\alpha(w) = F(w)$. This precisely says that $v \in V^{F/F_\alpha}$ i.e $F_\lambda(v) = F(v)$. If λ is a limit ordinal then $F_\lambda = \bigcup_{\alpha<\lambda} F_\lambda$ so $F_\lambda(v) = F(v)$ when $v \in \bigcup_{\alpha<\lambda} V_\alpha = V_\lambda$. Now since Q is left-rooted, $F = F_\lambda$ for some λ. This means that $\Phi(\mathscr{X})$ is $\oplus f_*(\mathscr{X})$-filtered. But any object in $\oplus f_*(\mathscr{X})$ is $f_*(\mathscr{X})$-filtered so we just insert such a filtration in each step as in [20, Lem. 1.6]. In this proof the objects form a set, but it is not necessarily for this particular statement. Indeed we always have $\operatorname{Filt}\operatorname{Filt}\mathscr{X} = \operatorname{Filt}\mathscr{X}$, for any class $\mathscr{X}$.

ii) When $\mathscr{X}$ is closed under filtrations we have

$$\operatorname{Filt} f_*(\mathscr{X}) \overset{\text{Rem. 2.6}}{\subseteq} \operatorname{Filt}\Phi(\mathscr{X}) \overset{\text{Lem. 2.8}}{\subseteq} \Phi(\operatorname{Filt}\mathscr{X}) \subseteq \Phi(\mathscr{X}).$$

iii) As $\operatorname{Filt} f_*(\mathscr{X})$ is closed under filtrations as mentioned we have

$$\begin{aligned}\operatorname{Filt}\Phi(\mathscr{X}) &\overset{\text{Lem. 2.8}}{\subseteq} \Phi(\operatorname{Filt}\mathscr{X}) \subseteq \operatorname{Filt} f_*(\operatorname{Filt}\mathscr{X})\\ &\overset{\text{Lem. 2.7}}{\subseteq} \operatorname{Filt}\operatorname{Filt} f_*(\mathscr{X}) \subseteq \operatorname{Filt} f_*(\mathscr{X})\\ &\overset{\text{Rem. 2.6}}{\subseteq} \operatorname{Filt}\Phi(\mathscr{X})\end{aligned}$$

iv) This is proven similar to *iii)*. Just observe that a filtration of summands is a summand of a filtration.

v) When $\mathscr{X}$ is $FP_{2.5}$ then $f_*(\mathscr{X})$ is $FP_{2.5}$ by Lemma 2.9 and so is $\operatorname{ext} f_*(\mathscr{X})$ by Lemma 1.3. Hence $\varinjlim \operatorname{ext} f_*(\mathscr{X})$ is closed under extensions by Proposition 1.1. We now have

$$\begin{aligned}\varinjlim \Phi(\mathscr{X}) &\overset{\text{Lem. 2.8}}{\subseteq} \Phi(\varinjlim \mathscr{X}) \subseteq \operatorname{Filt} f_*(\varinjlim \mathscr{X}) \overset{\text{Lem. 2.7}}{\subseteq} \operatorname{Filt} \varinjlim f_*(\mathscr{X})\\ &\subseteq \operatorname{Filt}\varinjlim \operatorname{ext} f_*(\mathscr{X}) \subseteq \varinjlim \operatorname{ext} f_*(\mathscr{X}) \overset{\text{Rem. 2.6}}{\subseteq} \varinjlim \operatorname{ext}\Phi(\mathscr{X})\\ &\overset{\text{Lem. 2.8}}{\subseteq} \varinjlim \Phi(\operatorname{ext}\mathscr{X}) \subseteq \varinjlim \Phi(\mathscr{X}).\end{aligned}$$

□

As mentioned in the introduction we also get *iii)* by combining results in [14] and [20] when $\mathscr{X}$ is a generating set and $\mathscr{A}$ is Grothendieck.

As a special case we get the known results from [10] and [7]:

Lemma 2.12. *Let $\mathscr{A}$ be an AB5-abelian category, let Q be a left-rooted quiver and let $\mathscr{X} \subseteq \mathscr{A}$ be a set of projective objects. Then*

i) $\Phi(\bigoplus X) = \bigoplus f_*(\mathscr{X}) = \bigoplus \Phi(\mathscr{X})$

ii) $\Phi(\operatorname{Add} X) = \operatorname{Add} f_*(\mathscr{X}) = \operatorname{Add}\Phi(\mathscr{X})$

If $\mathscr{A}$ has enough projectives, then

iii) $\Phi(\operatorname{Proj}\mathscr{A}) = \operatorname{Proj}(\operatorname{Rep}(Q,\mathscr{A}))$

If $\mathscr{A}$ is locally finitely presented, generated by $\operatorname{proj}(\mathscr{A})$ *(the finitely generated projective objects) then*

iv) $\Phi(\varinjlim \operatorname{proj}(\mathscr{A})) = \varinjlim \operatorname{proj}((\operatorname{Rep}(Q,\mathscr{A})))$

Proof. For *i)* and *ii)* just notice that any filtration is a sum as all extensions of projective objects are split. For *iii)* and *iv)* we notice that if $\mathscr{X} = Proj(\mathscr{A})$ (resp. $\mathscr{X} = \operatorname{proj}(\mathscr{A})$) generate $\mathscr{A}$ then $f_*(\mathscr{X}) \subseteq \operatorname{Proj}(\operatorname{Rep}(Q,\mathscr{A}))$ (resp. $f_*(\mathscr{X}) \subseteq \operatorname{proj}(\operatorname{Rep}(Q,\mathscr{A}))$ generate $\operatorname{Rep}(Q,\mathscr{A})$. Hence $\operatorname{Add} f_*(\mathscr{X}) = \operatorname{Proj}(\operatorname{Rep}(Q,\mathscr{A})$. (resp. $\operatorname{add} f_*(\mathscr{X}) = \operatorname{proj}(\operatorname{Rep}(Q,\mathscr{A}))$. Now use Theorem A *ii)* (resp. v)) □

As noted in the introduction, *iii)* can be seen by using cotorsion pairs as in [14]. In the rest of the paper we study the Gorenstein situation.

3. Gorenstein projective objects

We will now define the small (i.e. $FP_{2.5}$) Gorenstein projective objects and describe their direct limit closure using Theorem A.

Definition 3.1. Let $\mathscr{A}$ be an abelian category and $\mathscr{P}$ a class of objects in $\mathscr{A}$. A complete $\mathscr{P}$-resolution is an exact sequence with components in $\mathscr{P}$ that stays exact after applying $\mathrm{Hom}(P,-)$ and $\mathrm{Hom}(-,P)$ for any $P \in \mathscr{P}$.

We say that X has a complete $\mathscr{P}$-resolution if it is a syzygy in a complete $\mathscr{P}$-resolution, i.e. if there exists a complete $\mathscr{P}$-resolution

$$\ldots \to P_1 \to P_0 \to P_{-1} \to \ldots$$

s.t. $X = \ker(P_0 \to P_{-1})$.

We say that $X \in \mathscr{A}$ is Gorenstein projective (resp. Gorenstein injective) if it has a complete $\mathscr{P}$-resolution where $\mathscr{P}$ is the class of all projective (resp. injective) objects.

We let $\mathrm{GProj}(\mathscr{A})$ (resp. $\mathrm{GInj}(\mathscr{A})$)) denote the Gorenstein projective (resp. Gorenstein injective) objects of $\mathscr{A}$ and let $\mathrm{Gproj}(\mathscr{A}) = \mathrm{GProj}(\mathscr{A}) \cap FP_{2.5}(\mathscr{A})$.

Remark 3.2. Notice that the class $\mathrm{GProj}(\mathscr{A})$ is closed under extensions see [12, thm 2.5]. Hence so is $\mathrm{Gproj}(\mathscr{A})$ by Lemma 1.3.

Dually to the already mentioned characterization of the projective representations, we have a characterization of the injective representations. This was first noted in [8] and generalized to abelian categories in [14]. A similar description is possible for Gorenstein projective and Gorenstein injective objects as proven first for modules over Gorenstein rings in [8] and then modules over arbitrary rings in [11, Thm. 3.5.1]. This proof work in any abelian category with enough projective (resp. injective) objects. We collect the results here for ease of reference.

Theorem 3.3. *Let Q be a left-rooted quiver, $\mathscr{A}$ an abelian category with enough projective objects, and $\mathscr{B}$ a category with enough injective objects. Then*

$$\mathrm{Proj}(\mathrm{Rep}(Q,\mathscr{A})) = \Phi(\mathrm{Proj}(\mathscr{A}))$$
$$\mathrm{GProj}(\mathrm{Rep}(Q,\mathscr{A})) = \Phi(\mathrm{GProj}(\mathscr{A}))$$
$$\mathrm{Inj}(\mathrm{Rep}(Q^{\mathrm{op}},\mathscr{B})) = \Psi(\mathrm{Inj}(\mathscr{B}))$$
$$\mathrm{GInj}(\mathrm{Rep}(Q^{\mathrm{op}},\mathscr{B})) = \Psi(\mathrm{GInj}(\mathscr{B}))$$

where for $\mathscr{Y} \subseteq \mathscr{B}$ we define

$$\Psi(\mathscr{Y}) = \{F \in \mathrm{Rep}(Q^{\mathrm{op}},\mathscr{B}) \mid \forall v \in Q_0 : \psi_v^F \text{ epi and } \ker \psi_v^F \in \mathscr{X}\}$$

and

$$\psi_v^F = F(v) \to \prod_{v\to w} F(w).$$

As mentioned in the proofs, left-rooted is not needed for the inclusions ($\subseteq$) in the non-Gorenstein cases. We note that f_v preserves Gorenstein projectivity:

Lemma 3.4. *Suppose $\mathscr{A}$ satisfies AB4* or has enough projective objects or Q is locally path-finite. If $X \in \mathscr{A}$ is Gorenstein projective, then so is $f_v(X) \in \mathrm{Rep}(Q,\mathscr{A})$.*

Proof. Let $P_\bullet$ be a complete projective resolution of X. Then $f_v(P_\bullet)$ is exact and has projective components by Remark 2.4.

Obviously $\mathrm{Hom}(P, f_v(P_\bullet))$ is exact for any projective P, and $\mathrm{Hom}(f_v(P_\bullet), P) \cong \mathrm{Hom}(P_\bullet, e_v(P))$ is exact if e_v preserves projective objects.

If $\mathscr{A}$ has enough projective objects then $\mathrm{Proj}(\mathrm{Rep}(Q,\mathscr{A})) \subseteq \Phi(\mathrm{Proj}\mathscr{A})$. (Theorem 3.3) If $\mathscr{A}$ satisfies AB4* or Q is locally path-finite, then as in the proof of Lemma 2.9 e_v has an exact right-adjoint (see [14, 3.6]). In all cases e_v preserves projective objects. □

Using these and Theorem A we have

Proof of Theorem B. By Theorem A and Remark 3.2 we have

$$\varinjlim \Phi(\mathrm{Gproj}(\mathscr{A})) = \Phi(\varinjlim \mathrm{Gproj}(\mathscr{A})) = \varinjlim \mathrm{ext}\, f_*(\mathrm{Gproj}(\mathscr{A}))$$

Now f_v preserves smallness (Lemma 2.9 (i)) and Gorenstein projectivity (Lemma 3.4), so

$$\varinjlim \mathrm{ext}\, f_*(\mathrm{Gproj}(\mathscr{A})) \subseteq \varinjlim \mathrm{Gproj}(\mathrm{Rep}(Q,\mathscr{A})).$$

If Q is locally path-finite and target-finite, Theorem 3.3 and Lemma 2.9(iii) give

$$\mathrm{Gproj}(\mathrm{Rep}(Q,\mathscr{A})) = \Phi(\mathrm{GProj}(\mathscr{A})) \cap FP_{2.5}(\mathrm{Rep}(Q,\mathscr{A})) \subseteq \Phi(\mathrm{Gproj}(\mathscr{A}))$$

so

$$\varinjlim \mathrm{Gproj}(\mathrm{Rep}(Q,\mathscr{A})) \subseteq \varinjlim \Phi(\mathrm{Gproj}(\mathscr{A})).$$

If instead $\varinjlim \mathrm{Gproj}(\mathscr{A}) = \varinjlim \mathrm{GProj}(\mathscr{A})$ then by Theorem 3.3

$$\begin{aligned} \varinjlim \mathrm{Gproj}(\mathrm{Rep}(Q,\mathscr{A})) &\subseteq \varinjlim \mathrm{GProj}(\mathrm{Rep}(Q,\mathscr{A})) \\ &\subseteq \varinjlim \Phi(\mathrm{GProj}(\mathscr{A})) \\ &\subseteq \Phi(\varinjlim \mathrm{GProj}(\mathscr{A})) \\ &= \Phi(\varinjlim \mathrm{Gproj}(\mathscr{A})). \end{aligned}$$

□

4. Weakly Gorenstein flat objects

In this section we will first explain what we mean by an abstract Pontryagin dual. It mimics the behavior of $\mathrm{Ab}(-,\mathbb{Q}/\mathbb{Z})$. We will then define and describe the weakly Gorenstein flat objects and show when they equal $\varinjlim gP$.

Recall that a functor $F : \mathscr{C} \to \mathscr{D}$ *creates exactness* when $A \to B \to C$ is exact in $\mathscr{C}$ if and only if $FA \to FB \to FC$ is exact in $\mathscr{D}$.

Definition 4.1. A Pontryagin dual is a contravariant adjunction between abelian categories that creates exactness. I.e. let $\mathscr{C},\mathscr{D}$ be abelian categories. A *Pontryagin dual* between $\mathscr{C}$ and $\mathscr{D}$ consists of two functors

$$(-)^+ : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}, \quad (-)^+ : \mathscr{D}^{\mathrm{op}} \to \mathscr{C}$$

that both create exactness together with a natural ismorphism $\mathscr{C}(A, B^+) \cong \mathscr{D}(B, A^+)$.

We call it $\otimes$-*compatible* if there is a *continuous* bifunctor $\otimes : \mathscr{D} \times \mathscr{C} \to \mathscr{K}$ to some abelian category $\mathscr{K}$ s.t.

$$\mathscr{C}(A, B^+) \cong \mathscr{K}(B \otimes A, E) \cong \mathscr{D}(B, A^+)$$

for some injective cogenerator $E \in \mathscr{K}$ (i.e. $\mathscr{K}(-,E)$ creates exactness). Here continuous means that it respects direct limits.

Note that $\mathrm{Ab}(-,\mathbb{Q}/\mathbb{Z})\colon \mathrm{Ab}^{\mathrm{op}} \to \mathrm{Ab}$ is a Pontryagin dual compatible with the usual tensor product $\otimes\colon \mathrm{Ab} \times \mathrm{Ab} \to \mathrm{Ab}$ with $E = \mathbb{Q}/\mathbb{Z}$.

Example 4.2. As the following examples shows, (abstract) Pontyagin duals abound.

1) Let $(\mathscr{C}, [-,-], \otimes, 1)$ be a symmetric monoidal abelian category. Let $E \in \mathscr{C}$ be an injective cogenerator s.t. also $[-,E]$ creates exactness. Then $[-,E]$ is a $\otimes$-compatible Pontryagin dual, and any $\otimes$-compatible Pontryagin dual is of this form. It will thus also satisfy

$$[A, B^+] \cong (B \otimes A)^+ \cong [B, A^+].$$

This example includes the motivating example $\mathscr{C} = \mathrm{Ab}, E = \mathbb{Q}/\mathbb{Z}$ as well as $\mathscr{C} = \mathrm{Ch}(\mathrm{Ab}), E = \mathbb{Q}/\mathbb{Z}$ (i.e. $\mathbb{Q}/\mathbb{Z}$ in degree 0 and 0 otherwise).

2) If $(-)^+ : \mathscr{C}^{\text{op}} \to \mathscr{D}$ is a Pontryagin dual it induces a Potryagin dual $\text{Fun}(\mathscr{A},\mathscr{C})^{\text{op}} \to \text{Fun}(\mathscr{A}^{\text{op}},\mathscr{D})$ for any small category $\mathscr{A}$ by applying $(-)^+$ component-wise.

   If $(-)^+ : \mathscr{C}^{\text{op}} \to \mathscr{D}$ is compatible with $\otimes : \mathscr{D} \times \mathscr{C} \to \mathscr{K}$, then $(-)^+ : \text{Fun}(\mathscr{A},\mathscr{C})^{\text{op}} \to \text{Fun}(\mathscr{A}^{\text{op}},\mathscr{D})$ is compatible with $\otimes\colon \text{Fun}(\mathscr{A}^{\text{op}},\mathscr{D}) \times \text{Fun}(\mathscr{A},\mathscr{C}) \to \mathscr{K}$ where $G \otimes F$ is the coend of
$$\mathscr{A}^{\text{op}} \times \mathscr{A} \to \mathscr{D} \times \mathscr{C} \to \mathscr{K}$$
   i.e. the coequalizer of the two obvious maps
$$\bigoplus_{a\to b} G(b) \otimes F(a) \rightrightarrows \bigoplus_{a\in\mathscr{A}} G(a) \otimes F(a)$$
   provided the required colimits exists. (see Oberst and Röhrl [17] or Mac Lane [16, IX.6] for this construction).

   This includes the case $\text{Rep}(Q,\mathscr{C})$ for any quiver Q.
3) As in 2), any Pontryagin dual $(-)^+ : \mathscr{C}^{\text{op}} \to \mathscr{D}$ gives a component-wise Pontryagin dual $\text{Ch}(\mathscr{C})^{\text{op}} \to \text{Ch}(\mathscr{D})$ of chain-complexes. If $(-)^+ : \mathscr{C}^{\text{op}} \to \mathscr{D}$ is compatible with $\otimes : \mathscr{D} \otimes \mathscr{C} \to \mathscr{K}$ with injective cogenerator $E \in \mathscr{K}$, then $(-)^+ : \text{Ch}(\mathscr{C})^{\text{op}} \to \text{Ch}(\mathscr{D})$ is compatible with the total tensor product $\text{Ch}(\mathscr{D}) \times \text{Ch}(\mathscr{C}) \to \text{Ch}(\mathscr{K})$, the injective cogenerator beeing E in degree 0 and 0 otherwise.

   With $\mathscr{C} = \text{Ab}, (-)^+ = [-, \mathbb{Q}/\mathbb{Z}]$ this construction gives the standard one in Ch(Ab) as mentioned in 1).
4) If $\mathscr{C} = \mathscr{D}$ is symmetric monoidal with a $\otimes$-compatible Pontryagin dual as in 1) then the dual of a map $A \otimes X \overset{m}{\to} X$ gives a map $X^+ \otimes A \overset{m^+}{\to} X^+$ via the isomorphisms
$$\text{Hom}(X^+, (A \otimes X)^+) \cong \text{Hom}(X^+, [A, X^+]) \cong \text{Hom}(X^+ \otimes A, X^+).$$
   One can check that if A is a ring object and m is a left multiplication then m^+ is a right multiplication and we get a Pontryagin dual $(-)^+\colon (A\text{-Mod})^{\text{op}} \to \text{Mod-}A$ from the category of left A-modules to the category of right A-modules.

   This is $\otimes$-compatible with $-\otimes_A -\colon\ (\text{Mod-}A) \times (A\text{-Mod}) \to \mathscr{C}$, where $X \otimes_A Y$ is the coequalizer of the two obvious maps
$$X \otimes A \otimes Y \rightrightarrows X \otimes Y\ .$$
   (See Pareigis [18] for the details of this construction). This gives the standard Pontryagin dual in R-Mod for any ring R (i.e. a ring object in Ab), and by 3) the standard one in Ch(R-Mod). It also gives the character module of differential graded A-modules (DG-A-Mod) when A is a differential graded algebra, i.e. a ring object in Ch(Ab) (see Avramov, Foxby and Halperin [2]). By 2) we also get the one in [10, Cor 6.7] for $\text{Rep}(Q, R\text{-Mod})$ for any ring R and quiver Q.

Definition 4.3. Let $(-)^+ : \mathscr{C}^{\text{op}} \to \mathscr{D}$ be a Pontryagin dual. We say that

- $X \in \mathscr{C}$ is *flat* if X^+ is injective in $\mathscr{D}$,
- $X \in \mathscr{C}$ is *weakly Gorenstein flat* (wGFlat) if X^+ is Gorenstein injective.
- $F \in \text{Ch}(\mathscr{C})$ is a *complete flat resolution* if F^+ is a *complete injective resolution* in $\text{Ch}(\mathscr{D})$,
- $X \in \mathscr{C}$ is *Gorenstein flat* (GFlat) if it has a (i.e. is a syzygy in a) complete flat resolution,

Gorenstein flat always implies weakly Gorenstein flat. The other implication requires one to construct a complete flat resolution when the dual has a complete injective resolution. We will look at when this is possible in the next section.

With $\otimes$-compatibility these notions agree with the standard notions.

Proposition 4.4. *If $(-)^+ : \mathscr{C}^{\text{op}} \to \mathscr{D}$ is $\otimes$-compatible, then*

1) *$F \in \mathscr{C}$ is flat if and only if $- \otimes F$ is exact.*
2) *$F_\bullet$ is a complete flat resolution if and only if F_i is flat for all i and $I \otimes F_\bullet$ is exact for all injective objects $I \in \mathscr{D}$.*

Proof. 1) We have the following equivalences

$$\begin{aligned} F \text{ is flat} &\Leftrightarrow F^+ \text{ is injective} \\ &\Leftrightarrow \operatorname{Hom}(-, F^+) \text{ is exact} \\ &\Leftrightarrow \operatorname{Hom}(- \otimes F, E) \text{ is exact for some injective cogenerator } E \\ &\Leftrightarrow - \otimes F \text{ is exact} \end{aligned}$$

2) Let $F_\bullet \in \operatorname{Ch}(\mathscr{C})$. Then F_i^+ is injective iff F_i is flat and $\operatorname{Hom}(I, F_\bullet^+)$ is exact iff $\operatorname{Hom}(I \otimes F_\bullet, E)$ is exact for some injective cogenerator E by $\otimes$-compatibility, see Example 4.2 3). But this happens iff $I \otimes F_\bullet$ is exact. □

The following lemma shows how the classes $\Phi(\mathscr{X})$ (Definition 2.5) and $\Psi(\mathscr{X})$ (see Theorem 3.3) behave with respects to the Pontryagin duals. The proofs are straightforward.

Lemma 4.5. *Let $(-)^+ \colon \mathscr{A} \to \mathscr{B}$ be a Pontryagin dual between abelian categories, let Q a quiver, let $\mathscr{X} \subseteq \mathscr{A}$ and $\mathscr{Y} \subseteq \mathscr{B}$. Then*

$$\Phi(\mathscr{X})^+ \subset \Psi(\mathscr{X}^+).$$

In particular if $\mathscr{X} = \{X \in \mathscr{A} \mid X^+ \in \mathscr{Y}\}$ then

$$F \in \Phi(\mathscr{X}) \Leftrightarrow F^+ \in \Psi(\mathscr{Y}).$$

If Q is target-finite then

$$\Psi(\mathscr{Y})^+ \subseteq \Phi(\mathscr{Y}^+).$$

Proof. For the first assertion we must notice, that $(\phi_v^F)^+ = \psi_v^{F^+}$ for all $F \in \operatorname{Rep}(Q, \mathscr{A})$ and all $v \in Q_0$. For the second, that $(\psi_v^G)^+ = \phi_v^{G^+}$ for all $G \in \operatorname{Rep}(Q^{\mathrm{op}}, \mathscr{B})$ and all $v \in Q_0$ when Q is target-finite. This is because the product in the definition of $\psi_v^G \colon G(v) \to \prod_{v \to w \text{ in } Q^{\mathrm{op}}} G(w)$ is finite when Q^{op} is source-finite, thus it is a sum and so is the dual. □

This immediately gives the following:

Proof of Theorem C.

$$\begin{aligned} F \in \operatorname{Flat}(\operatorname{Rep}(Q, \mathscr{A})) &\overset{Def.\ 4.3}{\Longleftrightarrow} F^+ \in \operatorname{Inj}(\operatorname{Rep}(Q^{\mathrm{op}}, \mathscr{B})) \\ &\overset{Thm.\ 3.3}{\Longleftrightarrow} F^+ \in \Psi(\operatorname{Inj}(\mathscr{B})) \\ &\overset{Lem.\ 4.5}{\Longleftrightarrow} F \in \Phi(\operatorname{Flat}(\mathscr{A})) \end{aligned}$$

The same proof works in the Gorenstein situation. □

Remark 4.6. This gives a straightforward proof of [10, Thm 3.7] using the characterization of the injective representations from [8].

Combining this with Theorem B we get:

Corollary 4.7. *Let $(-)^+ : \mathscr{A}^{op} \to \mathscr{B}$ be a Pontryagin dual, let Q be a left-rooted quiver and assume*

- *$\mathscr{A}$ has enough projective objects*
- *$\mathscr{B}$ has enough injective objects*
- *Q is target-finite and locally path-finite, or $\varinjlim \operatorname{Gproj}(\mathscr{A}) = \varinjlim \operatorname{GProj}(\mathscr{A})$.*

If $\varinjlim \operatorname{Gproj} = \operatorname{wGFlat}$ in $\mathscr{A}$ then the same is true in $\operatorname{Rep}(Q, \mathscr{A})$.

5. Gorenstein flat objects

We will now find conditions on the category $\mathscr{A}$ and the quiver Q s.t.

$$\mathrm{wGFlat}(\mathrm{Rep}(Q, \mathscr{A})) = \mathrm{GFlat}(\mathrm{Rep}(Q, \mathscr{A})).$$

Firstly we have the following known result:

Proposition 5.1. *[12, Prop. 3.6] Let R be a right coherent ring. Then*

$$\mathrm{wGFlat}(\text{R-Mod}) = \mathrm{GFlat}(\text{R-Mod}).$$

Looking more closely at the proof (see Christensen [5, Thm. 6.4.2]) we arrive at Lemma 5.3. We include a full proof, as our notions of flatness differ.

Lemma 5.2. *Let $\mathscr{A}$ be an abelian category. If $0 \to X' \to J \to X \to 0$ is exact and J is injective (or just Gorenstein injective), X is Gorenstein injective and $\mathrm{Ext}^1(I, X') = 0$ for all injective $I \in \mathscr{A}$. Then X' is Gorenstein injective.*

Proof. This is the dual of [12, 2.11]. The proof is for modules but works in any abelian category. □

Now recall that a class $\mathscr{X} \subseteq \mathscr{C}$ is *preenveloping* if for every $M \in \mathscr{C}$ there is a map $\phi\colon M \to X$ called the *preenvelope* to some $X \in \mathscr{X}$ s.t. every map from M to an object in $\mathscr{X}$ factors through ϕ. It is monic whenever there exists some monomorphism from M to an object of $\mathscr{X}$.

Lemma 5.3. *Let $(-)^+ : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$ be a Pontryagin dual and assume*

(1) $\mathrm{Inj}(\mathscr{D})^+ \subseteq \mathrm{Flat}(\mathscr{C})$
(2') $\mathrm{Flat}(\mathscr{C})$ *is preenveloping.*
(3') $\mathscr{C}$ *has enough flat objects.*

Then any weakly Gorenstein flat object of $\mathscr{C}$ is Gorenstein flat.

Proof. Let X be weakly Gorenstein flat, i.e. X^+ is Gorenstein injective. Our goal is to construct a complete flat resolution for X. The left part of such a resolution is easy when $\mathscr{C}$ has enough flat objects. As X^+ is Gorenstein injective it has an injective resolution $I_\bullet$ s.t. $\mathrm{Hom}(J, I_\bullet)$ is exact for any injective J. But then this holds for any injective resolution of X^+. In particular $F_\bullet^+$, where $F_\bullet$ is a flat (left-) resolution of X which exists when $\mathscr{C}$ has enough flats.

For the right part we construct the resolution one piece at a time by constructing for any weakly Gorenstein flat $X \in \mathscr{C}$ a short exact sequence $0 \to X \to F \to X' \to 0$ where F is flat s.t. $\mathrm{Ext}^1(I, X'^+) = 0$ for any injective $I \in \mathscr{D}$. Then X'^+ is Gorenstein injective by Proposition 5.1 and this process can be continued to give a flat (right-) resolution $F_\bullet$ of X s.t. $\mathrm{Hom}(I, F_\bullet^+)$ is exact for any injective I.

So let again $X \in \mathscr{C}$ be weakly Gorenstein flat, and let $\varphi : X \to F$ be a flat preenvelope. We first show that ϕ is monic by showing that there exists some monomorphism from X to a flat object. Since X^+ is Gorenstein injective there exist an epimorphism $E \to X^+$ from some injective $E \in \mathscr{D}$. But then $X \to X^{++} \to E^+$ is monic, since $(-)^+$ creates exactness and $X^{+++} \to X^+$ is split epi by the unit-counit relation. Thus ϕ is monic since E^+ is flat by (1). We thus have a short exact sequence $0 \to X \xrightarrow{\phi} F \to X' \to 0$ inducing for any injective $I \in \mathscr{D}$ a long exact sequence

$$0 \to \mathrm{Hom}(I, X'^+) \to \mathrm{Hom}(I, F^+) \xrightarrow{\varphi^*} \mathrm{Hom}(I, X^+) \to \mathrm{Ext}^1(I, X'^+) \to \mathrm{Ext}^1(I, F^+).$$

Now φ^* is epi as $\varphi_* : \mathrm{Hom}(F, I^+) \to \mathrm{Hom}(X, I^+)$ is epi because I^+ is flat and φ is a flat preenvelope. Since $\mathrm{Ext}^1(I, F^+)$ is 0 because F^+ is injective we must have $\mathrm{Ext}^1(I, X'^+) = 0$. □

We notice that $\mathscr{A} = $ R-Mod satisfies these conditions when R is right coherent. ((1) is Xu [21, Lem. 3.1.4]) and (2') is [9, Prop. 6.5.1]). Our task is thus to find conditions on Q s.t. the conditions from Lemma 5.3 lift from $\mathscr{A}$ to $\mathrm{Rep}(Q, \mathscr{A})$. Lifting the condition that the flat objects are preenveloping is not obvious. But being closed under products is sometimes enough as the next lemma shows. We will reuse standard results on purity and therefore assume our Pontryagin Dual is $\otimes$-compatible and $\mathscr{A}$ to be generated by $\mathrm{proj}(\mathscr{A})$.

Lemma 5.4. *Let $\mathscr{A}$ be a locally finitely presented abelian category with a Pontryagin dual and assume that*

(2) The flat objects are closed under products
(3) $\mathscr{A}$ is generated by $\mathrm{proj}(\mathscr{A})$
(4) The Pontryagin dual is $\otimes$-compatible

Then the flat objects are preenveloping

Proof. Let $X \in \mathscr{A}$. The idea (as in [9, Prop. 6.2.1]) is to find a set of flat objects $\mathscr{S}$ s.t. every map $X \to Y$ with Y flat factors as $X \to S \hookrightarrow Y$ with $S \in \mathscr{S}$. Then we can construct a flat preenvelope as

$$X \to \prod_{\substack{S \in \mathscr{S} \\ \varphi : X \to S}} S_\varphi,$$

with $S_\varphi = S$ because the flat objects are closed under products by (2).

As in in the proof of [9, Lemma 5.3.12] there is a set of objects $\mathscr{S} \subseteq \mathscr{A}$ s.t. every map $X \to Y$ to some $Y \in \mathscr{A}$ factors as $X \to S \hookrightarrow Y$ for some $S \in \mathscr{S}$ with the property that, given a commutative square

$$\begin{array}{ccc} L_0 & \hookrightarrow & L_1 \\ \downarrow & \swarrow & \downarrow \\ S & \hookrightarrow & Y \end{array}$$

with L_0 finitely generated and L_1 finitely presented there is a lift $L_1 \to S$ s.t. the left triangle commutes. The proof is for modules and bounds size of $\mathscr{S}$ by some cardinality. If we are not interested in the cardinality, the proof works in any well-powered category, i.e. a category where there is only a set of subobjects of any given object. As in Adámek and and Rosický [1] any Grothendieck category is well-powered. We are left with proving that if Y is flat, so is S, i.e. if Y^+ is injective, so is S^+. Now Jensen and Lenzing [15, Prop. 7.16] shows (using (3)) that the above lifting property implies (in fact is equivalent to) that $S \hookrightarrow Y$ is a direct limit of split monomorphisms. [15, Thm 6.4] then shows (using (4)) that this implies, that $Y^+ \to S^+$ is split epi. (Equivalence of these statements uses that the generators in R-Mod are *dualizable*). Thus if Y^+ is injective, so is S^+. $\square$

Lemma 5.5. *Let $(-)^+ \colon \mathscr{A}^{\mathrm{op}} \to \mathscr{B}$ be a Pontryagin dual where $\mathscr{A}$ is AB4* and $\mathscr{B}$ has enough injective objects. Let Q be a left-rooted and target-finite quiver. If $\mathscr{A}$ satisfies* (1)-(4) *(from Lemma 5.3 and 5.4) then so does* $\mathrm{Rep}(Q, \mathscr{A})$.

Proof. For (1) we have

$$\begin{aligned} \mathrm{Inj}(\mathrm{Rep}(Q^{\mathrm{op}}, \mathscr{B}))^{++} &\overset{\text{Thm. 3.3}}{\subseteq} \Psi(\mathrm{Inj}(\mathscr{B}))^{++} \overset{\text{Lem. 4.5}}{\subseteq} \Psi(\mathrm{Inj}(\mathscr{B})^{++}) \\ &\subseteq \Psi(\mathrm{Inj}(\mathscr{B})) \overset{\text{Thm. 3.3}}{\subseteq} \mathrm{Inj}(\mathrm{Rep}(Q, \mathscr{B})) \end{aligned}$$

since $\mathscr{B}$ has enough injective objects and Q is left-rooted and target-finite. For (2) we have

$$\prod \operatorname{Flat}(\operatorname{Rep}(Q,\mathscr{A})) \overset{\text{Thm. C}}{\subseteq} \prod \Phi(\operatorname{Flat}\mathscr{A}) \overset{\text{Lem. 2.8}}{\subseteq} \Phi(\prod \operatorname{Flat}\mathscr{A})$$
$$\subseteq \Phi(\operatorname{Flat}\mathscr{A}) \overset{\text{Thm. C}}{\subseteq} \operatorname{Flat}(\operatorname{Rep}(Q,\mathscr{A})).$$

since $\mathscr{A}$ is AB4* and $\mathscr{B}$ has enough injective objects and Q is left-rooted and target-finite. (3) and (4) lifts without conditions on $\mathscr{A}$ and Q. For (3), if $\mathscr{A}$ is generated by a set $\mathscr{X}$ of finitely generated projective objects then $f_*(\mathscr{X})$ is a generating set of finitely generated projective objects by Lem. 2.9(i) and Remark 2.4. (4) is lifted in Example 4.2. □

Notice that $(3)-(4)$ holds for $\mathscr{A} = $ R-Mod over any ring R, and (2) is equivalent to R being right coherent [9, Prop. 3.2.24].

Proposition 5.6. *Let $\mathscr{A}$ be a locally finitely presented abelian AB4*-category. Let $\mathscr{B}$ be an abelian category with enough injective objects, let $(-)^+\colon \mathscr{A}^{\mathrm{op}} \to \mathscr{B}$ be a $\otimes$-compatible Pontryagin dual. If*

- *$\mathscr{A}$ is generated by* $\operatorname{proj}(\mathscr{A})$
- $\operatorname{Flat}(\mathscr{A})$ *is closed under products*
- $\operatorname{Inj}(\mathscr{B})^+ \subseteq \operatorname{Flat}(\mathscr{A})$

then $\operatorname{Flat}(\mathscr{A})$ *is preenveloping and* $\operatorname{wGFlat}(\mathscr{A}) = \operatorname{GFlat}(\mathscr{A})$. *Assume further that Q is a left-rooted and target-finite quiver. Then* $\operatorname{Flat}(\operatorname{Rep}(Q,\mathscr{A}))$ *is preenveloping and*

$$\operatorname{wGFlat}(\operatorname{Rep}(Q,\mathscr{A})) = \operatorname{GFlat}(\operatorname{Rep}(Q,\mathscr{A})).$$

Proof. This follows from Lemma 5.3 and 5.4 and 5.5. We also need (3') to hold and we could lift this directly by noting that f_v respects flatness, but it also follows from (3). □

We can now prove

Proof of Theorem D. Use Proposition 5.6 and the remark above it. □

Remark 5.7. In [8, Lem 6.9 and proof of Thm. 6.11] it is proved that

$$\operatorname{wGFlat}(\operatorname{Rep}(Q,\text{R-Mod})) = \operatorname{GFlat}((\operatorname{Rep}(Q,\text{R-Mod}))$$

when R is Iwanaga-Gorenstein and Q is only required to be left-rooted. Theorem D thus weakens the condition of R but must then strengthen the conditions on Q.

Corollary 5.8. *Let Q be a left-rooted quiver and let $\mathscr{A}$ be as in Proposition 5.6. If $\mathscr{A} = $ R-Mod for some Iwanaga-Gorenstein ring R or*

- $\varinjlim \operatorname{Gproj}(\mathscr{A}) = \operatorname{GFlat}(\mathscr{A})$ *and*
- *Q is target-finite and locally path-finite*

Then

$$\varinjlim \operatorname{Gproj}(\operatorname{Rep}(Q,\text{R-Mod}) = \operatorname{GFlat}(\operatorname{Rep}(Q,\text{R-Mod}) = \Phi(\operatorname{GFlat}(\text{R-Mod})).$$

Proof. Apply Corollary 4.7 and Proposition 5.6 (or Remark 5.7 for the Gorenstein case) to get $\varinjlim \operatorname{Gproj} = \operatorname{wGFlat} = \operatorname{GFlat}$ in $\operatorname{Rep}(Q,\mathscr{A})$. The last equality then follows from Theorem C. □

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Paper III

Computations of atom spectra

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COMPUTATIONS OF ATOM SPECTRA

RUNE HARDER BAK AND HENRIK HOLM

ABSTRACT. This is a contribution to the theory of atoms in abelian categories recently developed in a series of papers by Kanda. We present a method that enables one to explicitly compute the atom spectrum of the module category over a wide range of non-commutative rings. We illustrate our method and results by several examples.

1. INTRODUCTION

Building on works of Storrer [16], Kanda has, in a recent series of papers [10, 11, 12], developed the theory of atoms in abelian categories. The fundamental idea is to assign to every abelian category $\mathcal{A}$ the *atom spectrum*, denoted by $\operatorname{ASpec}\mathcal{A}$, in such a way that when $\Bbbk$ is a commutative ring, then $\operatorname{ASpec}(\Bbbk\text{-Mod})$ recovers the prime ideal spectrum $\operatorname{Spec}\Bbbk$. In Section 2 we recall a few basic definitions and facts from Kanda's theory.

Strong evidence suggests that Kanda's atom spectrum really is the "correct", and a very interesting, generalization of the prime ideal spectrum to abstract abelian categories. For example, in [10, Thm. 5.9] it is proved that for any locally noetherian Grothendieck category $\mathcal{A}$ there is a bijective correspondance between $\operatorname{ASpec}\mathcal{A}$ and isomorphism classes of indecomposable injective objects in $\mathcal{A}$. This is a generalization of Matlis' bijective correspondance between $\operatorname{Spec}\Bbbk$ and the set of isomorphism classes of indecomposable injective $\Bbbk$-modules over a commutative noetherian ring $\Bbbk$; see [15]. Furthermore, in [10, Thm. 5.5] it is shown that there are bijective correspondances between open subsets of $\operatorname{ASpec}\mathcal{A}$, Serre subcategories of $\operatorname{noeth}\mathcal{A}$, and localizing subcategories of $\mathcal{A}$. This generalizes Gabriel's bijective correspondances [6] between specialization-closed subsets of $\operatorname{Spec}\Bbbk$, Serre subcategories of $\Bbbk$-mod, and localizing subcategories of $\Bbbk$-Mod for a commutative noetherian ring $\Bbbk$. From a theoretical viewpoint, these results are very appealing, however, in the literature it seems that little effort has been put into actually computing the atom spectrum.

In this paper, we add value to the results mentioned above, and to other related results, by explicitly computing/describing the atom spectrum—not just as a set, but as a partially ordered set and as a topological space—of a wide range of abelian categories. Our main technical result, Theorem 3.7, shows that if $F_i\colon \mathcal{A}_i \to \mathcal{B}$ $(i \in I)$ is a family of functors between abelian categories satisfying suitable assumptions, then there is a homeomorphism and an order-isomorphism $f\colon \bigsqcup_{i\in I} \operatorname{ASpec}\mathcal{A}_i \to \operatorname{ASpec}\mathcal{B}$. Hence, if all the atom spectra $\operatorname{ASpec}\mathcal{A}_i$ are known, then so is $\operatorname{ASpec}\mathcal{B}$. One special case of this result is:

Theorem A. *Let $(Q,\mathcal{R})$ be a quiver with admissible relations and finitely many vertices. Let $\Bbbk Q$ be the path algebra of Q and consider the two-sided ideal $I = (\mathcal{R})$ in $\Bbbk Q$ generated by $\mathcal{R}$. There is an injective, continuous, open, and order-preserving map,*

$$\tilde{f}\colon \bigsqcup_{i\in Q_0} \operatorname{Spec}\Bbbk \longrightarrow \operatorname{ASpec}(\Bbbk Q/I\text{-Mod}) ,$$

given by (i^{th} copy of $\operatorname{Spec}\Bbbk$) $\ni \mathfrak{p} \longmapsto \langle \Bbbk Q/\tilde{\mathfrak{p}}(i)\rangle$. If, in addition, $(Q,\mathcal{R})$ is right rooted, then $\tilde{f}$ is also surjective, and hence it is a homeomorphism and an order-isomorphism.

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We prove Theorem A in Section 4, where we also give the definitions of admisible relations (4.3), right-rooted quivers (4.1), and of the ideals $\tilde{\mathfrak{p}}(i)$ (4.11). Note that in the terminology of Kanda [10, Def. 6.1], Theorem A yields that the *comonoform* left ideals in the ring $\Bbbk Q/I$ are precisely the ideals $\tilde{\mathfrak{p}}(i)/I$ where $\mathfrak{p}$ is a prime ideal in $\Bbbk$ and i is a vertex in Q.

Theorem A applies e.g. to show that for every $n,m \geqslant 1$ the map

$$\operatorname{Spec}\Bbbk \longrightarrow \operatorname{ASpec}(\Bbbk\langle x_1,\dots,x_n\rangle/(x_1,\dots,x_n)^m\text{-Mod}) \quad \text{given by} \quad \mathfrak{p} \longmapsto \langle \Bbbk/\mathfrak{p}\rangle$$

is a homeomorphism and an order-isomorphism; see Example 4.14. Actually, Theorem A is a special case of Theorem 4.9 which yields a homeomorphism and an order-isomorphism $\operatorname{ASpec}(\operatorname{Rep}((Q,\mathcal{R}),\mathcal{A})) \cong \bigsqcup_{i\in Q_0} \operatorname{ASpec}\mathcal{A}$ for every right rooted quiver $(Q,\mathcal{R})$ with admissible relations (Q may have infinitely many vertices) and any $\Bbbk$-linear abelian category $\mathcal{A}$. From this stronger result one gets e.g. $\operatorname{ASpec}(\operatorname{Ch}\mathcal{A}) \cong \bigsqcup_{i\in\mathbb{Z}} \operatorname{ASpec}\mathcal{A}$; see Example 4.10.

In Section 5 we apply the previously mentioned (technical/abstract) Theorem 3.7 to compute the atom spectrum of comma categories:

Theorem B. *Let $\mathcal{A} \xrightarrow{U} \mathcal{C} \xleftarrow{V} \mathcal{B}$ be functors between abelian categories, where U has a right adjoint and V is left exact. Let $(U\downarrow V)$ be the associated comma category. There is a homeomorphism and an order-isomorphism,*

$$f\colon \operatorname{ASpec}\mathcal{A} \sqcup \operatorname{ASpec}\mathcal{B} \xrightarrow{\sim} \operatorname{ASpec}(U\downarrow V)\,,$$

given by $\langle H\rangle \longmapsto \langle S_{\mathcal{A}}H\rangle$ for $\langle H\rangle \in \operatorname{ASpec}\mathcal{A}$ and $\langle H\rangle \longmapsto \langle S_{\mathcal{B}}H\rangle$ for $\langle H\rangle \in \operatorname{ASpec}\mathcal{B}$.

Theorem B applies e.g. to show that for the non-commutative ring

$$T = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix},$$

where A and B are commutative rings and $M = {}_BM_A$ is a (B,A)-bimodule, there is a homeomorphism and an order-isomorphism $\operatorname{Spec}A \sqcup \operatorname{Spec}B \longrightarrow \operatorname{ASpec}(T\text{-Mod})$ given by

$$\operatorname{Spec}A \ni \mathfrak{p} \longmapsto \left\langle T\Big/\begin{pmatrix} \mathfrak{p} & 0 \\ M & B \end{pmatrix}\right\rangle \quad \text{and} \quad \operatorname{Spec}B \ni \mathfrak{q} \longmapsto \left\langle T\Big/\begin{pmatrix} A & 0 \\ M & \mathfrak{q} \end{pmatrix}\right\rangle;$$

see Example 5.4 for details.

We end the paper with Appendix A where we present some background material on representations of quivers with relations that is needed, and taken for granted, in Section 4.

2. Kanda's theory of atoms

We recall a few definitions and results from Kanda's theory [10, 11, 12] of atoms.

2.1. Let $\mathcal{A}$ be an abelian category. An object $H \in \mathcal{A}$ is called *monoform* if $H \neq 0$ and for every non-zero subobject $N \rightarrowtail H$ there exists no common non-zero subobject of H and H/N, i.e. if there exist monomorphisms $H \leftarrowtail X \rightarrowtail H/N$ in $\mathcal{A}$, then $X = 0$. See [10, Def. 2.1].

Two monoform objects H and H' in $\mathcal{A}$ are said to be *atom equivalent* if there exists a common non-zero subobject of H and H'. Atom equivalence is an equivalence relation on the collection of monoform objects; the equivalence class of a monoform object H is denoted by $\langle H\rangle$ and is called an *atom* in $\mathcal{A}$. The collection of all atoms in $\mathcal{A}$ is called the *atom spectrum* of $\mathcal{A}$ and denoted by $\operatorname{ASpec}\mathcal{A}$. See [10, Def. 2.7, Prop. 2.8, and Def. 3.1].

The atom spectrum of an abelian category comes equipped with a topology and a partial order which we now explain.

2.2. The *atom support* of an object $M \in \mathcal{A}$ is defined in [10, Def. 3.2] and is given by

$$\operatorname{ASupp}M = \left\{ \langle H\rangle \in \operatorname{ASpec}\mathcal{A} \;\middle|\; \begin{array}{l} H \text{ is a monoform object such that} \\ H \cong L/L' \text{ for some } L' \subseteq L \subseteq M \end{array} \right\}.$$

A subset $\Phi \subseteq \operatorname{ASpec}\mathcal{A}$ is said to be *open* if for every $\langle H\rangle \in \Phi$ there exists $H' \in \langle H\rangle$ such that $\operatorname{ASupp} H' \subseteq \Phi$. The collection of open subsets defines a topology, called the *localization topology*, on $\operatorname{ASpec}\mathcal{A}$, see [10, Def. 3.7 and Prop. 3.8], and the collection

$$\{\operatorname{ASupp} M \mid M \in \mathcal{A}\}$$

is an open basis of this topology; see [12, Prop. 3.2].

2.3. The topological space $\operatorname{ASpec}\mathcal{A}$ is a so-called *Kolmogorov space* (or a T_0-*space*), see [12, Prop. 3.5], and any such space X can be equipped with a canonical partial order $\leqslant$, called the *specialization order*, where $x \leqslant y$ means that $x \in \overline{\{y\}}$ (the closure of $\{y\}$ in X). This partial order on $\operatorname{ASpec}\mathcal{A}$ is more explicitly described in [12, Prop. 4.2].

2.4 Lemma. *Let X and Y be Kolmogorov spaces equipped with their specialization orders. Any continuous map $f\colon X \to Y$ is automatically order-preserving.*

Proof. Assume that $x \leqslant y$ in X, that is, $x \in \overline{\{y\}}$. Then $f(x) \in f(\overline{\{y\}}) \subseteq \overline{f(\{y\})} = \overline{\{f(y)\}}$, where the inclusion holds as f is continuous, and thus $f(x) \leqslant f(y)$ in Y. □

2.5. For a commutative ring $\Bbbk$, its prime ideal spectrum coincides with the atom spectrum of its module category in the following sense: By [10, Props. 6.2, 7.1, and 7.2(1)], see also [16, p. 631], there is a bijection of sets:

$$q\colon \operatorname{Spec}\Bbbk \longrightarrow \operatorname{ASpec}(\Bbbk\text{-Mod}) \quad \text{given by} \quad \mathfrak{p} \longmapsto \langle \Bbbk/\mathfrak{p}\rangle\,.$$

This bijection is even an order-isomorphism between the partially ordered set $(\operatorname{Spec}\Bbbk, \subseteq)$ and $\operatorname{ASpec}(\Bbbk\text{-Mod})$ equipped with its specialization order; see [12, Prop. 4.3]. Via q the open subsets of $\operatorname{ASpec}(\Bbbk\text{-Mod})$ correspond to the *specialization-closed* subsets of $\operatorname{Spec}\Bbbk$; see [10, Prop. 7.2(2)]. In this paper, we always consider $\operatorname{Spec}\Bbbk$ as a partially ordered set w.r.t. to inclusion and as a topological space in which the open sets are the specialization-closed ones. In this way, the map q above is an order-isomorphism and a homeomorphism.*

3. The main result

In this section, we explain how a suitably nice functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories induces a map $\operatorname{ASpec}F\colon \operatorname{ASpec}\mathcal{A} \to \operatorname{ASpec}\mathcal{B}$. The terminology in the following definition is inspired by a similar terminology from Diers [3, Chap. 1.8], where it is defined what it means for a functor to lift direct factors.

3.1 Definition. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor. We say that F *lifts subobjects* if for any $A \in \mathcal{A}$ and any monomorphism $\iota\colon B \rightarrowtail FA$ in $\mathcal{B}$ there exist a monomorphism $\iota'\colon A' \rightarrowtail A$ in $\mathcal{A}$ and an isomorphism $B \xrightarrow{\cong} FA'$ such that the following diagram commutes:

$$\begin{array}{ccc} B & \overset{\iota}{\rightarrowtail} & FA \\ & {\scriptstyle\cong}\searrow \quad \nearrow{\scriptstyle F\iota'} & \\ & FA'\,. & \end{array}$$

(We will usually suppress the isomorphism and treat it as an equality $B = FA'$.)

3.2 Remark. Recall that any full and faithful (= fully faithful) functor $F\colon \mathcal{A} \to \mathcal{B}$ is injective on objects up to isomorphism, that is, if $FA \cong FA'$ in $\mathcal{B}$, then $A \cong A'$ in $\mathcal{A}$.

3.3 Observation. Let M be an object in $\mathcal{A}$. If $\langle H\rangle \in \operatorname{ASupp} M$, then $\operatorname{ASupp} H \subseteq \operatorname{ASupp} M$. Indeed, if $H \cong L/L'$ for some $L' \subseteq L \subseteq M$, then [10, Prop. 3.3] applied to the two sequences $0 \to L' \to L \to H \to 0$ and $0 \to L \to M \to M/L \to 0$ yield $\operatorname{ASupp} H \subseteq \operatorname{ASupp} L \subseteq \operatorname{ASupp} M$.

* We emphasize that q is not a homeomorphism when $\operatorname{Spec}\Bbbk$ is equipped with the (usual) Zariski topology! In the case where $\Bbbk$ is noetherian, the topological $\operatorname{Spec}\Bbbk$ considered by us and Kanda [10] is the *Hochster dual*, in the sense of [8, Prop. 8], of the spectral space $\operatorname{Spec}\Bbbk$ with Zariski topology.

3.4 Proposition. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a full, faithful, and exact functor between abelian categories that lifts subobjects. There is a well-defined map,*

$$\operatorname{ASpec} F\colon \operatorname{ASpec}\mathcal{A} \longrightarrow \operatorname{ASpec}\mathcal{B} \quad \textit{given by} \quad \langle H\rangle \longmapsto \langle FH\rangle\,,$$

which is injective, continuous, open, and order-preserving.

Proof. First we argue that for any object $H \in \mathcal{A}$ we have:

$$H \text{ is monoform (in } \mathcal{A}) \iff FH \text{ is monoform (in } \mathcal{B})\,. \qquad (\sharp 1)$$

"$\Leftarrow$": Assume that FH is monoform. By definition, FH is non-zero, so H must be non-zero as well. Let M be a non-zero subobject of H and assume that there are monomorphisms $H \leftarrowtail X \rightarrowtail H/M$. We must prove that $X = 0$. As F is exact we get monomorphisms $FH \leftarrowtail FX \rightarrowtail F(H/M) \cong (FH)/(FM)$. Note that $FM \neq 0$ by Remark 3.2, so it follows that $FX = 0$ since FH is monoform. Hence $X = 0$, as desired.

"$\Rightarrow$": Assume that H is monoform. As $H \neq 0$ we have $FH \neq 0$ by Remark 3.2. Let N be a non-zero subobject of FH and let $FH \leftarrowtail Y \rightarrowtail (FH)/N$ be monomorphisms. We must prove that $Y = 0$. Since F lifts subobjects, the monomorphism $N \rightarrowtail FH$ is the image under F of a monomorphisms $M \rightarrowtail H$. As $FM = N$ is non-zero, so is M. Since F is exact, the canonical morphism $(FH)/N = (FH)/(FM) \to F(H/M)$ is an isomorphism. By precomposing this isomorphism with $Y \rightarrowtail (FH)/N$ we get a monomorphism $Y \rightarrowtail F(H/M)$, which is then the image under F of some monomorphism $X \rightarrowtail H/M$. The monomorphism $FH \leftarrowtail Y$ is also the image of a monomorphism $H \leftarrowtail X'$, and since $FX = Y = FX'$ we have $X \cong X'$ by Remark 3.2. Hence there are monomorphisms $H \leftarrowtail X \rightarrowtail H/M$, and as H is monoform we conclude that $X = 0$. Hence $Y = FX = 0$, as desired.

Next note that if H and H' are monoform objects in $\mathcal{A}$ which are atom equivalent, i.e. they have a common non-zero subobject M, then FM is a common non-zero subobject of FH and FH', and hence FH and FH' are atom equivalent monoform objects in $\mathcal{B}$. This, together with the implication "$\Rightarrow$" in ($\sharp$1), shows that the map $\operatorname{ASpec} F$ is well-defined.

To see that $\operatorname{ASpec} F$ is injective, let H and H' be monoform objects in $\mathcal{A}$ for which FH and FH' are atom equivalent, i.e. there is a common non-zero subobject $FH \leftarrowtail N \rightarrowtail FH'$. From the fact that F lifts subobjects, and from Remark 3.2, we get that these monomorphisms are the images under F of monomorphisms $H \leftarrowtail M \rightarrowtail H'$. As $FM = N$ is non-zero, so if M. Thus, H and H' are atom equivalent in $\mathcal{A}$.

Next we show that for every object $M \in \mathcal{A}$ there is an equality:

$$(\operatorname{ASpec} F)(\operatorname{ASupp} M) = \operatorname{ASupp} FM\,. \qquad (\sharp 2)$$

"$\subseteq$": Let $\langle H\rangle \in \operatorname{ASupp} M$, that is, $H \cong L/L'$ for some $L' \subseteq L \subseteq M$. As the functor F is exact we have $FL' \subseteq FL \subseteq FM$ and $FH \cong F(L/L') \cong (FL)/(FL')$ and hence the element $\langle FH\rangle = (\operatorname{ASpec} F)(\langle H\rangle)$ belongs to $\operatorname{ASupp} FM$.

"$\supseteq$": Let $\langle I\rangle \in \operatorname{ASupp} FM$, that is, I is a monoform object in $\mathcal{B}$ with $I \cong N/N'$ for some $N' \subseteq N \subseteq FM$. Since $N \subseteq FM$ and F lifts subobjects, there is a subobject $L \subseteq M$ with $FL = N$. Similarly, as $N' \subseteq N = FL$ there is a subobject $L' \subseteq L$ with $FL' = N'$. We now have $L' \subseteq L \subseteq M$ and $F(L/L') \cong (FL)/(FL') \cong N/N' \cong I$. Since I is monoform, we conclude from ($\sharp$1) that the object $H := L/L'$ is monoform, so $\langle H\rangle$ belongs to $\operatorname{ASupp} M$. And by constuction, $(\operatorname{ASpec} F)(\langle H\rangle) = \langle FH\rangle = \langle I\rangle$.

Recall from 2.2 that $\{\operatorname{ASupp} M \mid M \in \mathcal{A}\}$ is an open basis of the topology on $\operatorname{ASpec}\mathcal{A}$ (and similarly for $\operatorname{ASpec}\mathcal{B}$). It is therefore evident from ($\sharp$2) that $\operatorname{ASpec} F$ is an open map.

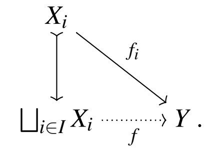
Furthermore, to show that $\operatorname{ASpec} F$ is continuous, it suffices to show that for any $N \in \mathcal{B}$, the set $\Phi := (\operatorname{ASpec} F)^{-1}(\operatorname{ASupp} N)$ is open in $\operatorname{ASpec}\mathcal{A}$. For every $\langle H\rangle \in \Phi$ we have

$$\operatorname{ASupp} H \subseteq (\operatorname{ASpec} F)^{-1}(\operatorname{ASupp} FH) \subseteq (\operatorname{ASpec} F)^{-1}(\operatorname{ASupp} N) = \Phi\,,$$

where the first inclusion follows from ($\sharp$2) and the second one follows from Observation 3.3 since $\langle FH\rangle \in \operatorname{ASupp} N$. Hence $\langle H\rangle \in \operatorname{ASupp} H \subseteq \Phi$, so Φ is open by 2.2.

From the continuity and from Lemma 2.4 we get that $\operatorname{ASpec} F$ is order-preserving. $\square$

3.5. Let $\{X_i\}_{i\in I}$ be a family of sets and write $\coprod_{i\in I} X_i$ for the disjoint union. This is the coproduct of $\{X_i\}_{i\in I}$ in the category of sets, so given any family $f_i\colon X_i \to Y$ of maps, there is a unique map f that makes the following diagram commute:

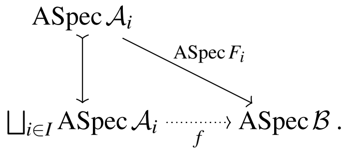


In the case where each X_i is a topological space, $\coprod_{i\in I} X_i$ is equipped with the disjoint union topology, and this yields the coproduct of $\{X_i\}_{i\in I}$ in the category of topological spaces. In fact, for the maps f_i and f in the diagram above, it is well-known that one has:

$$f \text{ is continuous (open)} \iff f_i \text{ is continuous (open) for every } i \in I\,.$$

If each X_i is a Kolmogorov space, then so is $\coprod_{i\in I} X_i$ (and hence it is the coproduct in the category of Kolmogorov spaces). In this case, and if Y is also a Kolmogorov space, any continuous map f in the diagram above is automatically order-preserving by Lemma 2.4.

3.6 Proposition. *Let $F_i\colon \mathcal{A}_i \to \mathcal{B}$ $(i \in I)$ be a family of full, faithful, and exact functors between abelian categories that lift subobjects. There exists a unique map f that makes the following diagram commute:*



That is, f maps $\langle H\rangle \in \operatorname{ASpec}\mathcal{A}_i$ to $\langle F_iH\rangle \in \operatorname{ASpec}\mathcal{B}$. This map f is continuous, open, and order-preserving.

Proof. Immediate from Proposition 3.4 and 3.5. □

Our next goal is to find conditions on the functors F_i from Proposition 3.6 which ensure that the map f is bijective, and hence a homeomorphism and an order-isomorphism.

3.7 Theorem. *Let $F_i\colon \mathcal{A}_i \to \mathcal{B}$ $(i \in I)$ be a family of functors as in Proposition 3.6 and consider the induced continuous, open, and order-preserving map*

$$f\colon \coprod_{i\in I} \operatorname{ASpec}\mathcal{A}_i \longrightarrow \operatorname{ASpec}\mathcal{B}\,.$$

The map f is injective provided that the following condition holds:

(a) *For $i \neq j$ and $A_i \in \mathcal{A}_i$ and $A_j \in \mathcal{A}_j$ the only common subobject of F_iA_i and F_jA_j is* 0.

The map f is surjective provided that each F_i has a right adjoint G_i satisfying:

(b) *For every $B \neq 0$ in $\mathcal{B}$ there exists $i \in I$ with $G_iB \neq 0$.*

(c) *For every $i \in I$ and $B \in \mathcal{B}$ the counit $F_iG_iB \to B$ is monic.*

Thus, if (a), (b), *and* (c) *hold, then f is a homeomorphism and an order-isomorphism.*

Proof. First we show that condition (a) implies injectivity of f. Let $\langle H\rangle \in \operatorname{ASpec}\mathcal{A}_i$ and $\langle H'\rangle \in \operatorname{ASpec}\mathcal{A}_j$ be arbitrary elements in $\coprod_{i\in I}\operatorname{ASpec}\mathcal{A}_i$ with $f(\langle H\rangle) = f(\langle H'\rangle)$, that is, $\langle F_iH\rangle = \langle F_jH'\rangle$. This means that the monoform objects F_iH and F_jH' are atom equivalent, so they contain a common non-zero subobject N. By the assumption (a), we must have $i = j$. As the map $\operatorname{ASpec}F_i$ is injective, see Proposition 3.4, we conclude that $\langle H\rangle = \langle H'\rangle$.

Next we show that conditions (b) and (c) imply surjectivity of f. Let H be any monoform object in $\mathcal{B}$. Since $H \neq 0$ there exists by (b) some $i \in I$ with $G_iH \neq 0$. This implies $F_iG_iH \neq 0$, see Remark 3.2, so F_iG_iH is a non-zero subobject of the monoform object H by (c). Thus [10, Prop. 2.2] implies that F_iG_iH is a monoform object, atom equivalent to

H. From (♯1) we get that G_iH is monoform, so $\langle G_iH\rangle$ is an element in ASpec $\mathcal{A}_i$ satisfying $f(\langle G_iH\rangle) = \langle F_iG_iH\rangle = \langle H\rangle$. □

4. Application to quiver representations

In this section, we will apply Theorem 3.7 to compute the atom spectrum of the category of $\mathcal{A}$-valued representations of any (well-behaved) quiver with relations $(Q,\mathcal{R})$. Here $\mathcal{A}$ is a $\Bbbk$-linear abelian category and $\Bbbk$ is any commutative ring. Appendix A contains some background material on quivers with relations and their representations needed in this section. The main result is Theorem 4.9, and we also prove Theorem A from the Introduction.

Enochs, Estrada, and García Rozas define in [4, Sect. 4] what it means for a quiver, *without* relations, to be right rooted. Below we extend their definition to quivers *with* relations. To parse the following, recall the notion of the $\Bbbk$-linearization of a category and that of an ideal in a $\Bbbk$-linear category, as described in A.2 and A.3.

4.1 Definition. A quiver with relations $(Q,\mathcal{R})$ is said to be *right rooted* if for every infinite sequence of (not necessarily different) composable arrows in Q,

$$\bullet \xrightarrow{a_1} \bullet \xrightarrow{a_2} \bullet \xrightarrow{a_3} \cdots,$$

there exists $N \in \mathbb{N}$ such that the path $a_N \cdots a_1$ (which is a morphism in the category $\Bbbk\bar{Q}$) belongs to the two-sided ideal $(\mathcal{R}) \subseteq \Bbbk\bar{Q}$.

4.2 Observation. Let $(Q,\mathcal{R})$ be a quiver with relations. If there exists *no* infinite sequence $\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$ of (not necessarily different) composable arrows in Q, then $(Q,\mathcal{R})$ is right rooted, as the requirement in Definition 4.1 becomes void. If Q is a quiver without relations, i.e. $\mathcal{R} = \emptyset$ and hence $(\mathcal{R}) = \{0\}$, then Q is right rooted if and only if there exists no such infinite sequence $\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$; indeed, a path $a_N \cdots a_1$ is never zero in the absence of relations. Consequently, our Definition 4.1 of right rootedness for quivers with relations extends the similar definition for quivers without relations found in [4, Sect. 4].

Next we introduce admissible relations and stalk functors.

4.3 Definition. A relation ρ in a quiver Q, see A.3, is called *admissible* if the coefficient in the linear combination ρ to every trivial path e_i ($i \in Q_0$) is zero. We refer to a quiver with relations $(Q,\mathcal{R})$ as a *quiver with admissible relations* if every relation in $\mathcal{R}$ is admissible.

As we shall be interested in right rooted quivers with admissible relations, it seems in order to compare these notions with the more classic notion of admissibility:

4.4 Remark. According to [1, Chap. II.2 Def. 2.1], a set $\mathcal{R}$ of relations in a quiver Q with finitely many vertices is *admissible* if $\mathfrak{a}^m \subseteq (\mathcal{R}) \subseteq \mathfrak{a}^2$ holds for some $m \geqslant 2$. Here $\mathfrak{a}$ is the *arrow ideal* in $\Bbbk Q$, that is, the two-sided ideal generated by all arrows in Q. Note that:

$\mathcal{R}$ is admissible as in [1, Chap. II.2 Def. 2.1]	$\Longrightarrow$	$\mathcal{R}$ is admissible as in Definition 4.3 and $(Q,\mathcal{R})$ is right rooted as in Definition 4.1.

Indeed, in terms of the arrow ideal, our definition of admissibility simply means $(\mathcal{R}) \subseteq \mathfrak{a}$[†], and if there is an inclusion $\mathfrak{a}^m \subseteq (\mathcal{R})$, then Definition 4.1 holds with (universal) $N = m$.

If $(Q,\mathcal{R})$ is right rooted, one does not necessarily have $\mathfrak{a}^m \subseteq (\mathcal{R})$ for some m. Indeed, let Q be a quiver with one vertex and countably many loops $x_1, x_2, \ldots$. For each $\ell \geqslant 1$ let $\mathcal{R}_\ell = \{x_{u_\ell} \cdots x_{u_1} x_\ell \,|\, u_1, \ldots, u_\ell \in \mathbb{N}\}$ be the set of all paths of length $\ell + 1$ starting with x_ℓ.

[†] Often, not much interesting comes from considering relations in $\mathfrak{a} \setminus \mathfrak{a}^2$. To illustrate this point, consider e.g. the Kronecker quiver $K_2 = \bullet \overset{a}{\underset{b}{\rightrightarrows}} \bullet$ with one relation $\rho := a - b \in \mathfrak{a} = (a,b) \subseteq \Bbbk K_2$. Clearly, the category $\operatorname{Rep}((K_2,\{\rho\}),\mathcal{A})$ is equivalent to $\operatorname{Rep}(A_2,\mathcal{A})$ where $A_2 = \bullet \longrightarrow \bullet$. So the representation theory of $(K_2,\{\rho\})$ is already covered by the representation theory of a quiver (in this case, A_2) with relations (in this case, $\mathcal{R} = \emptyset$) contained in the square of the arrow ideal.

Set $\mathcal{R} = \bigcup_{\ell \geqslant 1} \mathcal{R}_\ell$. Evidently, $(\mathcal{R}) \subseteq \mathfrak{a}^2 = (x_1, x_2, \ldots)^2$ and $(Q,\mathcal{R})$ is right rooted. As none of the elements $x_1, x_2^2, x_3^3, \ldots$ belong to $(\mathcal{R})$ we have $\mathfrak{a}^m \nsubseteq (\mathcal{R})$ for every m.

However, if Q has only finitely many arrows (in addition to having only finitely many vertices), then right rootedness of $(Q,\mathcal{R})$ means precisely that $\mathfrak{a}^m \subseteq (\mathcal{R})$ for some m.

4.5 Definition. Let Q be a quiver and let $\mathcal{A}$ be an abelian category. For every $i \in Q_0$ there is a *stalk functor* $S_i \colon \mathcal{A} \to \operatorname{Rep}(Q,\mathcal{A})$ which assigns to $A \in \mathcal{A}$ the *stalk representation* S_iA given by $(S_iA)(j) = 0$ for every vertex $j \neq i$ in Q_0 and $(S_iA)(i) = A$. For every path $p \neq e_i$ in Q one has $(S_iA)(p) = 0$ and, of course, $(S_iA)(e_i) = \mathrm{id}_A$.

4.6 Remark. Let ρ be a relation in a quiver Q and let $x_i \in \Bbbk$ be the coefficient (which may or may not be zero) to the path e_i in the linear combination ρ. If A is any object in a $\Bbbk$-linear abelian category $\mathcal{A}$, then $(S_iA)(\rho) = x_i\,\mathrm{id}_A$. It follows that the stalk representation S_iA satisfies every admissible relation. Thus, if $(Q,\mathcal{R})$ be a quiver with admissible relations, then every S_i can be viewed as a functor $\mathcal{A} \to \operatorname{Rep}((Q,\mathcal{R}),\mathcal{A})$.

4.7. Let $(Q,\mathcal{R})$ be a quiver with admissible relations and let $\mathcal{A}$ be a $\Bbbk$-linear abelian category. For every $i \in Q_0$ the stalk functor $S_i \colon \mathcal{A} \to \operatorname{Rep}((Q,\mathcal{R}),\mathcal{A})$ from Remark 4.6 has a right adjoint, namely the functor $K_i \colon \operatorname{Rep}((Q,\mathcal{R}),\mathcal{A}) \to \mathcal{A}$ given by

$$K_iX = \bigcap_{a\colon i \to j} \operatorname{Ker} X(a) = \operatorname{Ker}\Big(X(i) \xrightarrow{\psi_i^X} \prod_{a\colon i \to j} X(j) \Big)\,,$$

where the intersection/product is taken over all arrows $a\colon i \to j$ in Q with source i, and ψ_i^X is the morphism whose a^{th} coordinate function is $X(a)\colon X(i) \to X(j)$. For a quiver without relations ($\mathcal{R} = \emptyset$), the adjunctions (S_i, K_i) were established in [9, Thm. 4.5], but evidently this also works for quivers with admissible relations.

Note that the existence of K_i requires that the product $\prod_{a\colon i \to j}$ can be formed in $\mathcal{A}$; this is the case if, for example, $\mathcal{A}$ is complete (satisfies AB3*) or if $\mathcal{A}$ is arbitrary but there are only finitely many arrows in Q with source i. We tacitly assume that each K_i exists.

4.8 Lemma. *Let $(Q,\mathcal{R})$ be a quiver with admissible relations, let $\mathcal{A}$ be a $\Bbbk$-linear abelian category, and let $X \in \operatorname{Rep}((Q,\mathcal{R}),\mathcal{A})$. If $X \neq 0$ and $K_iX = 0$ holds for all $i \in Q_0$, then there exists an infinite sequence of (not necessarily different) composable arrows in Q,*

$$\bullet \xrightarrow{a_1} \bullet \xrightarrow{a_2} \bullet \xrightarrow{a_3} \cdots\,, \qquad (\sharp 3)$$

such that $X(a_n \cdots a_1) \neq 0$ for every $n \geqslant 1$. In particular, if $(Q,\mathcal{R})$ is right rooted and $X \neq 0$, then $K_iX \neq 0$ for some $i \in Q_0$.

Proof. As $X \neq 0$ we have $X(i_1) \neq 0$ for some vertex i_1. As $K_{i_1}X = 0$ we have $X(i_1) \nsubseteq K_{i_1}X$ so there is at least one arrow $a_1 \colon i_1 \to i_2$ with $X(i_1) \nsubseteq \operatorname{Ker} X(a_1)$, and hence $X(a_1) \neq 0$. As $0 \neq \operatorname{Im} X(a_1) \subseteq X(i_2)$ and $K_{i_2}X = 0$ we have $\operatorname{Im} X(a_1) \nsubseteq K_{i_2}X$, so there is at least one arrow $a_2 \colon i_2 \to i_3$ such that $\operatorname{Im} X(a_1) \nsubseteq \operatorname{Ker} X(a_2)$. This means that $X(a_2) \circ X(a_1) = X(a_2a_1)$ is non-zero. Continuing in this manner, the first assertion in the lemma follows.

For the second assertion, assume that there is some $X \neq 0$ with $K_iX = 0$ for all $i \in Q_0$. By the first assertion there exists an infinite sequence of composable arrows ($\sharp$3) such that $X(a_n \cdots a_1) \neq 0$ for every $n \geqslant 1$. Hence $a_n \cdots a_1 \notin (\mathcal{R}) \subseteq \Bbbk\vec{Q}$ holds for every $n \geqslant 1$ by the lower equivalence in the diagram in A.3 Thus $(Q,\mathcal{R})$ is not right rooted. ☐

The result below shows that for a *right rooted* quiver with *admissible* relations $(Q,\mathcal{R})$, the atom spectrum of $\operatorname{Rep}((Q,\mathcal{R}),\mathcal{A})$ depends only on the atom spectrum of $\mathcal{A}$ and on the (cardinal) number of vertices in Q. The arrows and the relations in Q play no (further) role!

4.9 Theorem. *Let $(Q,\mathcal{R})$ be a quiver with admissible relations and let $\mathcal{A}$ be any $\Bbbk$-linear abelian category. There is an injective, continuous, open, and order-preserving map,*

$$f\colon \textstyle\bigsqcup_{i \in Q_0} \operatorname{ASpec} \mathcal{A} \longrightarrow \operatorname{ASpec}(\operatorname{Rep}((Q,\mathcal{R}),\mathcal{A}))\,,$$

given by (i^{th} copy of $\operatorname{ASpec}\mathcal{A}$) $\ni \langle H\rangle \longmapsto \langle S_iH\rangle$. *If, in addition,* $(Q,\mathcal{R})$ *is right rooted, then* f *is also surjective, and hence it is a homeomorphism and an order-isomorphism.*

Proof. We apply Theorem 3.7 to the functors $F_i = S_i$ and $G_i = K_i$ ($i \in Q_0$) from 4.5 and 4.7. The functor S_i is obviously exact, and it also lifts subobjects as every subobject of S_iA has the form S_iA' for a subobject $A' \rightarrowtail A$ in $\mathcal{A}$. It is immediate from the definitions that the unit $\mathrm{id}_{\mathcal{A}} \to K_iS_i$ of the adjunction (S_i,K_i) is an isomorphism, and hence S_i is full and faithful by (the dual of) [14, IV.3, Thm. 1]. Hence the functors S_i meet the requirements in Proposition 3.6 and we get that f is well-defined, continuous, open, and order-preserving.

Evidently condition (a) in Theorem 3.7 holds, so f is injective. Now assume that $(Q,\mathcal{R})$ is right rooted. To prove that f is surjective we verify conditions (b) and (c) in Theorem 3.7. Condition (b) holds by Lemma 4.8. For every representation X the counit $S_iK_iX \to X$ is monic, that is, $(S_iK_iX)(j) \to X(j)$ is monic for every $j \in Q_0$. Indeed, for $j \neq i$ this is clear as $(S_iK_iX)(j) = 0$; and for $j = i$ we have $(S_iK_iX)(i) = K_iX = \bigcap_{a\colon i\to j}\operatorname{Ker}X(a)$, which is a subobject of $X(i)$. Hence (c) holds as well. □

4.10 Example. The quiver (without relations):

$$A^{\infty}_{\infty}:\quad \cdots \longrightarrow \underset{2}{\bullet} \xrightarrow{d_2} \underset{1}{\bullet} \xrightarrow{d_1} \underset{0}{\bullet} \xrightarrow{d_0} \underset{-1}{\bullet} \xrightarrow{d_{-1}} \underset{-2}{\bullet} \xrightarrow{d_{-2}} \cdots$$

is not right rooted, but when equipped with the admissible relations $\mathcal{R} = \{d_{n-1}d_n \,|\, n \in \mathbb{Z}\}$ it becomes right rooted. For any ($\mathbb{Z}$-linear) abelian category $\mathcal{A}$, the category $\operatorname{Rep}((A^{\infty}_{\infty},\mathcal{R}),\mathcal{A})$ is equivalent to the category $\operatorname{Ch}\mathcal{A}$ of chain complexes in $\mathcal{A}$. Hence Theorem 4.9 yields a homeomorphism and an order-isomorphism

$$\textstyle\bigsqcup_{i\in\mathbb{Z}} \operatorname{ASpec}\mathcal{A} \longrightarrow \operatorname{ASpec}(\operatorname{Ch}\mathcal{A})$$

given by (i^{th} copy of $\operatorname{ASpec}\mathcal{A}$) $\ni \langle H\rangle \longmapsto \langle \cdots \to 0 \to 0 \to H \to 0 \to 0 \to \cdots\rangle$ with H in degree i and zero in all other degrees.

The next goal is to apply Theorem 4.9 to prove Theorem A from the Introduction.

4.11 Definition. Let Q be a quiver with finitely many vertices. For every ideal $\mathfrak{p}$ in $\Bbbk$ and every vertex i in Q set $\tilde{\mathfrak{p}}(i) = \{\xi \in \Bbbk Q \,|\, \text{the coefficient to } e_i \text{ in } \xi \text{ belongs to } \mathfrak{p}\}$.

4.12 Lemma. *With the notation above, the set* $\tilde{\mathfrak{p}}(i)$ *is a two-sided ideal in* $\Bbbk Q$ *which contains every admissible relation.*

Proof. Let $p \neq e_i$ be a path in Q and let ξ be an element in $\Bbbk Q$. In the linear combinations $p\xi$ and ξp the coefficient to e_i is zero. In the linear combinations $e_i\xi$ and ξe_i the coefficient to e_i is the same as the coefficient to e_i in the given element ξ. Hence $\tilde{\mathfrak{p}}(i)$ is a two-sided ideal in $\Bbbk Q$. By Definition 4.3, every admissible relation belongs to $\tilde{\mathfrak{p}}(i)$. □

Proof of Theorem A. Let $\tilde{f}$ be the map defined by commutativity of the diagram:

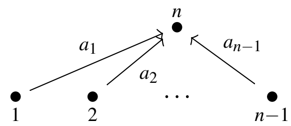
$$\begin{array}{ccc} \bigsqcup_{i\in Q_0}\operatorname{Spec}\Bbbk & \xrightarrow{\ \tilde{f}\ } & \operatorname{ASpec}(\Bbbk Q/I\text{-Mod}) \\ {\scriptstyle \bigsqcup_{i\in Q_0} q}\ \big\downarrow\ \wr & & \wr\ \big\uparrow\ {\scriptstyle \operatorname{ASpec}U} \\ \bigsqcup_{i\in Q_0}\operatorname{ASpec}(\Bbbk\text{-Mod}) & \xrightarrow{\ f\ } & \operatorname{ASpec}(\operatorname{Rep}((Q,\mathcal{R}),\Bbbk\text{-Mod}))\,. \end{array} \qquad (\sharp 4)$$

Here the lower horizontal map is the map from Theorem 4.9 with $\mathcal{A} = \Bbbk$-Mod; the left vertical map is the order-isomorphism and homeomorphism described in 2.5; and the right vertical map is induced by the equivalence of categories $U\colon \operatorname{Rep}((Q,\mathcal{R}),\Bbbk\text{-Mod}) \to \Bbbk Q/I\text{-Mod}$ given in A.4. An element $\mathfrak{p} \in$ (i^{th} copy of $\operatorname{Spec}\Bbbk$) is by $\bigsqcup_{i\in Q_0} q$ mapped to the atom $\langle \Bbbk/\mathfrak{p}\rangle \in$ (i^{th} copy of $\operatorname{ASpec}(\Bbbk\text{-Mod})$), which by f is mapped to the atom $\langle S_i(\Bbbk/\mathfrak{p})\rangle$. The functor U maps the representation $S_i(\Bbbk/\mathfrak{p})$ to the left $\Bbbk Q/I$-module (= a left $\Bbbk Q$-module killed by I) whose underlying $\Bbbk$-module is $\Bbbk/\mathfrak{p}$ (more precisely, $0\oplus\cdots\oplus 0\oplus\Bbbk/\mathfrak{p}\oplus 0\oplus\cdots\oplus 0$ with a "0" for each vertex $\neq i$) on which e_i acts as the identity and $p\cdot\Bbbk/\mathfrak{p}=0$ for all paths $p\neq e_i$. This means that the left $\Bbbk Q/I$-module $U(S_i(\Bbbk/\mathfrak{p}))$ is isomorphic to $\Bbbk Q/\tilde{\mathfrak{p}}(i)$. Indeed, $\Bbbk Q/\tilde{\mathfrak{p}}(i)$ *is* a $\Bbbk Q/I$-module as $I\subseteq\tilde{\mathfrak{p}}(i)$ by Lemma 4.12; and as a $\Bbbk$-module it is isomorphic to $\Bbbk/\mathfrak{p}$ since the $\Bbbk$-linear map $\Bbbk Q\to\Bbbk/\mathfrak{p}$ given by $\xi\mapsto$ [(coefficient to e_i in ξ)]$_{\mathfrak{p}}$ has kernel $\tilde{\mathfrak{p}}(i)$. As noted in the proof of Lemma 4.12, every path $p\neq e_i$ multiplies $\Bbbk Q$ into $\tilde{\mathfrak{p}}(i)$, so one has $p\cdot\Bbbk Q/\tilde{\mathfrak{p}}(i)=0$, and e_i acts as the identity on $\Bbbk Q/\tilde{\mathfrak{p}}(i)$. Having proved the isomorphism $U(S_i(\Bbbk/\mathfrak{p}))\cong\Bbbk Q/\tilde{\mathfrak{p}}(i)$, it follows that $\tilde{f}(\mathfrak{p})=\langle\Bbbk Q/\tilde{\mathfrak{p}}(i)\rangle$. Thus $\tilde{f}$ acts as described in the theorem. The assertions about $\tilde{f}$ follow from the commutative diagram (♯4) and from the properties of the map f given in Theorem 4.9. □

Below we examine the map $\tilde{f}$ from Theorem A in a some concrete of examples.

4.13 Example. Consider the $(n-1)$-subspace quiver (no relations), which is right rooted:

Σ_n :



The path algebra $\Bbbk\Sigma_n$ is isomorphic to the following $\Bbbk$-subalgebra of $\mathrm{M}_n(\Bbbk)$:

$$\mathrm{L}_n(\Bbbk)=\{\,(x_{ij})\in\mathrm{M}_n(\Bbbk)\mid x_{ij}=0\text{ if }i\neq n\text{ and }i\neq j\,\}\,.$$

Under this isomorphism the arrow a_i in Σ_n corresponds to the matrix $\alpha_i\in\mathrm{L}_n(\Bbbk)$ with 1 in entry (n,i) and 0 elsewhere, and the trivial path e_i corresponds to the matrix $\varepsilon_i\in\mathrm{L}_n(\Bbbk)$ with 1 in entry (i,i) and 0 elsewhere. It follows that, via this isomorphism, the ideal $\tilde{\mathfrak{p}}(i)\subseteq\Bbbk\Sigma_n$ from Definition 4.11 is identified with the ideal

$$\tilde{\mathfrak{p}}(i)=\begin{pmatrix}\Bbbk & & & \\ & \ddots & & 0 \\ 0 & & \mathfrak{p} & \\ & & & \ddots \\ \Bbbk & \cdots & \Bbbk & \cdots & \Bbbk\end{pmatrix}\subseteq\mathrm{L}_n(\Bbbk)\,.$$

Now Theorem A yields a homeomorphism and an order-isomorphism,

$$\textstyle\bigsqcup_{i=1}^{n}\operatorname{Spec}\Bbbk\longrightarrow\operatorname{ASpec}(\mathrm{L}_n(\Bbbk)\text{-Mod})$$

given by (i^{th} copy of $\operatorname{Spec}\Bbbk$) $\ni\mathfrak{p}\longmapsto\langle\mathrm{L}_n(\Bbbk)/\tilde{\mathfrak{p}}(i)\rangle$.

4.14 Example. Let Q be any quiver with finitely many vertices. Let $m>0$ be any natural number and let $\mathcal{P}_m$ be the relations consisting of all paths in Q of length m. Clearly these relations are admissible and $(Q,\mathcal{P}_m)$ is right rooted. If $\mathfrak{a}$ denotes the arrow ideal in $\Bbbk Q$, then $(\mathcal{P}_m)=\mathfrak{a}^m$, so Theorem A yields a homeomorphism and an order-isomorphism,

$$\textstyle\bigsqcup_{i\in Q_0}\operatorname{Spec}\Bbbk\longrightarrow\operatorname{ASpec}(\Bbbk Q/\mathfrak{a}^m\text{-Mod})\,,$$

given by (i^{th} copy of $\operatorname{Spec}\Bbbk$) $\ni\mathfrak{p}\longmapsto\langle\Bbbk Q/\tilde{\mathfrak{p}}(i)\rangle$. In the special case where Q is the quiver with one vertex and n loops $x_1,\dots,x_n$ one has $\Bbbk Q=\Bbbk\langle x_1,\dots,x_n\rangle$, the free $\Bbbk$-algebra. Moreover, for $\mathfrak{p}\in\operatorname{Spec}\Bbbk$ we have $\tilde{\mathfrak{p}}=\mathfrak{p}+(x_1,\dots,x_n)$ and hence $\Bbbk Q/\tilde{\mathfrak{p}}\cong\Bbbk/\mathfrak{p}$, which is a module over $\Bbbk\langle x_1,\dots,x_n\rangle$ where all the variables $x_1,\dots,x_n$ act as zero. Thus the map

$$\operatorname{Spec}\Bbbk\longrightarrow\operatorname{ASpec}(\Bbbk\langle x_1,\dots,x_n\rangle/(x_1,\dots,x_n)^m\text{-Mod})$$

given by $\operatorname{Spec}\Bbbk\ni\mathfrak{p}\longmapsto\langle\Bbbk/\mathfrak{p}\rangle$ is a homeomorphism and an order-isomorphism.

We end with an example illustrating the necessity of the assumptions in Theorem A. We shall see that the map $\tilde{f}$ need not be surjective if $(Q,\mathcal{R})$ is not right rooted and that the situation is more subtle when the relations are not admissible.

4.15 Example. Consider the Jordan quiver (which is not right rooted):

$$J: \quad \bullet \circlearrowleft X$$

The path algebra $\Bbbk J$ is isomorphic to the polynomial ring $\Bbbk[X]$, which is commutative, so via the homeomorphism and order-isomorphism $q\colon \operatorname{Spec}\Bbbk[X] \to \operatorname{ASpec}(\Bbbk[X]\text{-Mod})$ in 2.5, the map $\tilde{f}\colon \operatorname{Spec}\Bbbk \to \operatorname{ASpec}(\Bbbk[X]\text{-Mod})$ from Theorem A may be identified with a map

$$\operatorname{Spec}\Bbbk \longrightarrow \operatorname{Spec}\Bbbk[X]\,.$$

It is not hard to see that this map is given by $\mathfrak{p} \mapsto \mathfrak{p} + (X) = \{f(X) \in \Bbbk[X] \,|\, f(0) \in \mathfrak{p}\}$, so it is injective but not surjective. Typical prime ideals in $\Bbbk[X]$ that are not of the form $\mathfrak{p} + (X)$ are $\mathfrak{q}[X]$ where $\mathfrak{q} \in \operatorname{Spec}\Bbbk$. Also notice that for the Jordan quiver, the functor from 4.7,

$$\Bbbk[X]\text{-Mod} \simeq \operatorname{Rep}(J, \Bbbk\text{-Mod}) \xrightarrow{\;K\;} \Bbbk\text{-Mod}\,,$$

maps a $\Bbbk[X]$-module M to $KM = \operatorname{Ker}\big(M \xrightarrow{X\cdot} M\big)$. Thus it may happen that $KM = 0$ (if multiplication by X on M is injective) even though $M \neq 0$. This also shows that the last assertion in Lemma 4.8 can fail for quivers that are not right rooted.

Now let $\Bbbk = \mathbb{Z}$ and consider e.g. the relations $\mathcal{R} = \{X^3, 2\}$ in J (where "2" means two times the trivial path on the unique vertex in J). Then $(J, \mathcal{R})$ is right rooted because of the relation X^3, however, the relation 2 is not admissible. In this case,

$$\operatorname{Rep}((J,\mathcal{R}), \mathbb{Z}\text{-Mod}) \cong \mathbb{Z}[X]/(X^3,2)\text{-Mod} = \mathbb{F}_2[X]/(X^3)\text{-Mod}\,,$$

so $\operatorname{ASpec}(\operatorname{Rep}((J,\mathcal{R}),\mathbb{Z}\text{-Mod}))$ consists of a single element. This set is not even equipotent to $\operatorname{Spec}\mathbb{Z}$, in particular, there exists no homeomorphism or order-isomorphism between $\operatorname{ASpec}(\operatorname{Rep}((J,\mathcal{R}),\mathbb{Z}\text{-Mod}))$ and $\operatorname{Spec}\mathbb{Z}$.

5. Application to comma categories

In this section, we consider the *comma category* $(U \downarrow V)$, see [14, II.6], associated to a pair of additive functors between abelian categories:

$$\mathcal{A} \xrightarrow{\;U\;} \mathcal{C} \xleftarrow{\;V\;} \mathcal{B}\,.$$

An object in $(U \downarrow V)$ is a triple (A, B, θ) where $A \in \mathcal{A}$, $B \in \mathcal{B}$ are objects and $\theta\colon UA \to VB$ is a morphism in $\mathcal{C}$. A morphism $(A,B,\theta) \to (A',B',\theta')$ in $(U \downarrow V)$ is a pair of morphisms (α,β), where $\alpha\colon A \to A'$ is a morphism in $\mathcal{A}$ and $\beta\colon B \to B'$ is a morphism in $\mathcal{B}$, such that the following diagram commutes:

$$\begin{array}{ccc} UA & \xrightarrow{\;U\alpha\;} & UA' \\ {\scriptstyle\theta}\big\downarrow & & \big\downarrow{\scriptstyle\theta'} \\ VB & \xrightarrow{\;V\beta\;} & VB'\,. \end{array}$$

The comma category arising from the special case $\mathcal{A} \xrightarrow{\;U\;} \mathcal{B} \xleftarrow{\;\mathrm{id}_{\mathcal{B}}\;} \mathcal{B}$ is written $(U \downarrow \mathcal{B})$.

The notion and the theory of atoms only make sense in abelian categories. In general, the comma category is *not* abelian—not even if the categories $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are abelian and the functors U and V are additive, as we have assumed. However, under weak assumptions, $(U \downarrow V)$ *is* abelian, as we now prove. Two special cases of the following result can be found in [5, Prop. 1.1 and remarks on p. 6], namely the cases where U or V is the identity functor.

5.1 Proposition. *If U is right exact and V is left exact, then $(U \downarrow V)$ is abelian.*

Proof. It is straightforward to see that $(U \downarrow V)$ is an additive category.

We now show that every morphim $(\alpha,\beta)\colon (A,B,\theta) \to (A',B',\theta')$ in $(U \downarrow V)$ has a kernel. Let $\kappa\colon K \to A$ be a kernel of α and let $\lambda\colon L \to B$ be a kernel of β. As V is left exact, the morphism $V\lambda\colon VL \to VB$ is a kernel of $V\beta$, so there is a (unique) morphism ϑ that makes the following diagram commute:

$$\begin{array}{ccccccc} & & UK & \xrightarrow{U\kappa} & UA & \xrightarrow{U\alpha} & UA' \\ & & \Big\downarrow{\scriptstyle\vartheta} & & \Big\downarrow{\scriptstyle\theta} & & \Big\downarrow{\scriptstyle\theta'} \\ 0 & \longrightarrow & VL & \xrightarrow{V\lambda} & VB & \xrightarrow{V\beta} & VB' . \end{array} \tag{♯5}$$

We claim that $(\kappa,\lambda)\colon (K,L,\vartheta) \to (A,B,\theta)$ is a kernel of (α,β). By construction, the composition $(\alpha,\beta)\circ(\kappa,\lambda)$ is zero. Let $(\kappa',\lambda')\colon (K',L',\vartheta') \to (A,B,\theta)$ be any morphism in $(U\!\downarrow\!V)$ such that $(\alpha,\beta)\circ(\kappa',\lambda')$ is zero. We must show that (κ',λ') factors uniquely through (κ,λ).

Note that we have unique factorizations $K' \xrightarrow{\varphi} K \xrightarrow{\kappa} A$ of κ' and $L' \xrightarrow{\psi} L \xrightarrow{\lambda} B$ of λ' by the universal property of kernels since $\alpha\kappa' = 0$ and $\beta\lambda' = 0$. From these factorizations, the commutativity of (♯5), and from the fact that (κ',λ') is a morphism in $(U\!\downarrow\!V)$, we get:

$$V\lambda \circ \vartheta \circ U\varphi = \theta \circ U\kappa \circ U\varphi = \theta \circ U\kappa' = V\lambda' \circ \vartheta' = V\lambda \circ V\psi \circ \vartheta' .$$

As $V\lambda$ is monic we conclude that $\vartheta \circ U\varphi = V\psi \circ \vartheta'$, so $(\varphi,\psi)\colon (K',L',\vartheta') \to (K,L,\vartheta)$ is a morphism in $(U\!\downarrow\!V)$ with $(\kappa,\lambda)\circ(\varphi,\psi) = (\kappa',\lambda')$, that is, (κ',λ') factors through (κ,λ).

A similar argument shows that every morphism in $(U\!\downarrow\!V)$ has a cokernel; this uses the assumed right exactness of U. As for kernels, cokernels are computed componentwise.

Next we show that every monomorphism $(\alpha,\beta)\colon (A,B,\theta) \to (A',B',\theta')$ in $(U\!\downarrow\!V)$ is a kernel. We have just shown that (α,β) has a kernel, namely (K,L,ϑ) where K is a kernel of α and L is a kernel of β. Thus, if (α,β) is monic, then (K,L,ϑ) is forced to be zero, so α and β must both be monic. Let $0 \longrightarrow A \xrightarrow{\alpha} A' \xrightarrow{\rho} C \longrightarrow 0$ and $0 \longrightarrow B \xrightarrow{\beta} B' \xrightarrow{\sigma} D \longrightarrow 0$ be short exact sequences in $\mathcal{A}$ and $\mathcal{B}$. From the componentwise constructions of kernels and cokernels in $(U\!\downarrow\!V)$ given above, it follows that (ρ,σ) is a morphism in $(U\!\downarrow\!V)$ whose kernel is precisely the given monomorphism (α,β).

A similar argument shows that every epimorphism in $(U\!\downarrow\!V)$ is a cokernel. □

5.2 Definition. As for quiver representations, see Definition 4.5, there are *stalk functors*,

$$\mathcal{A} \xrightarrow{S_{\mathcal{A}}} (U\!\downarrow\!V) \xleftarrow{S_{\mathcal{B}}} \mathcal{B} ,$$

defined by $S_{\mathcal{A}}\colon A \longmapsto (A, 0, UA \xrightarrow{0} V0)$ and $S_{\mathcal{B}}\colon B \longmapsto (0, B, U0 \xrightarrow{0} VB)$.

We now describe the right adjoints of these stalk functors.

5.3 Lemma. *The following asertions hold.*

(a) *The stalk functor $S_{\mathcal{B}}$ has a right adjoint $K_{\mathcal{B}}\colon (U\!\downarrow\!V) \to \mathcal{B}$ given by $(X,Y,\theta) \mapsto Y$.*

(b) *Assume that U has a right adjoint $U^!$ and let η be the unit of the adjunction. The stalk functor $S_{\mathcal{A}}$ has a right adjoint $K_{\mathcal{A}}\colon (U\!\downarrow\!V) \to \mathcal{A}$ given by $(X,Y,\theta) \mapsto \mathrm{Ker}(U^!\theta \circ \eta_X)$, i.e. the kernel of the morphism $X \xrightarrow{\eta_X} U^!UX \xrightarrow{U^!\theta} U^!UY$.*

In particular, if an object (X,Y,θ) in $(U\!\downarrow\!V)$ satisfies $K_{\mathcal{A}}(X,Y,\theta) = 0$ and $K_{\mathcal{B}}(X,Y,\theta) = 0$, then one has $(X,Y,\theta) = 0$.

Proof. (a): Let $B \in \mathcal{B}$ and $(X,Y,\theta) \in (U\!\downarrow\!V)$ be objects. It is immediate from Definition 5.2 that a morphism $S_{\mathcal{B}}(B) \to (X,Y,\theta)$ in $(U\!\downarrow\!V)$ is the same as a morphism $\beta\colon B \to Y$ in $\mathcal{B}$.

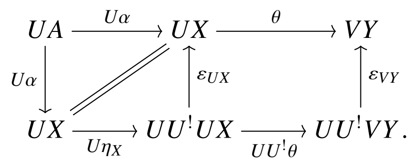
(b): Write η and ε for the unit and counit of the assumed adjunction $(U,U^!)$. Let $A \in \mathcal{A}$ and $(X,Y,\theta) \in (U\!\downarrow\!V)$ be objects. It is immediate from Definition 5.2 that a morphism $S_{\mathcal{A}}(A) \to (X,Y,\theta)$ in $(U\!\downarrow\!V)$ is the same as a morphism $\alpha\colon A \to X$ in $\mathcal{A}$ such that the composition $\theta \circ U\alpha\colon UA \to VY$ is zero. We claim that $\theta \circ U\alpha = 0$ if and only if $U^!\theta \circ \eta_X \circ \alpha = 0$. Indeed, the "only if" part follows directly from the identities

$$U^!\theta \circ \eta_X \circ \alpha = U^!\theta \circ U^!U\alpha \circ \eta_A = U^!(\theta \circ U\alpha) \circ \eta_A ,$$

where the first equality holds by naturality of η. The "if" part follows from the identities

$$\theta \circ U\alpha \;=\; \theta \circ \varepsilon_{UX} \circ U\eta_X \circ U\alpha \;=\; \varepsilon_{VY} \circ UU^!\theta \circ U\eta_X \circ U\alpha \;=\; \varepsilon_{VY} \circ U(U^!\theta \circ \eta_X \circ \alpha)\,,$$

where the first equality is by the unit-counit relation [14, IV.1 Thm. 1(ii)] and the second is by naturality of ε. This is illustrated in the following diagram:



Therefore, a morphism $S_{\mathcal{A}}(A) \to (X,Y,\theta)$ in $(U\!\downarrow\! V)$ is the same as a morphism $\alpha\colon A \to X$ in $\mathcal{A}$ with $U^!\theta \circ \eta_X \circ \alpha = 0$, and by the universal property of the kernel, such morphisms are in bijective correspondance with morphisms $A \to \operatorname{Ker}(U^!\theta \circ \eta_X)$. This proves (b).

For the last statement, note that $K_{\mathcal{B}}(X,Y,\theta) = 0$ yields $Y = 0$. Thus θ is the zero morphism $UX \to 0$ and consequently $U^!\theta \circ \eta_X$ is the zero morphism $X \to 0$. It follows that $X = K_{\mathcal{A}}(X,Y,\theta) = 0$, so $(X,Y,\theta) = 0$ in $(U\!\downarrow\! V)$. □

We are now in a position to show Theorem B from the Introduction.

Proof of Theorem B. First note that the under the given assumptions, the comma category $(U\!\downarrow\! V)$ is abelian by Proposition 5.1, so it makes sense to consider its atom spectrum. We will apply Theorem 3.7 to the functors $S_{\mathcal{A}}$ and $S_{\mathcal{B}}$ from Definition 5.2 whose right adjoints are $K_{\mathcal{A}}$ and $K_{\mathcal{B}}$ from Lemma 5.3. As shown in the proof of Proposition 5.1, kernels and cokernels in $(U\!\downarrow\! V)$ are computed componentwise, so the functor $S_{\mathcal{A}}$ is exact. It also lifts subobjects as every subobject of $S_{\mathcal{A}}(A)$ has the form $S_{\mathcal{A}}(A')$ for a subobject $A' \rightarrowtail A$ in $\mathcal{A}$. It is clear from the definitions that the unit $\mathrm{id}_{\mathcal{A}} \to K_{\mathcal{A}}S_{\mathcal{A}}$ of the adjunction $(S_{\mathcal{A}}, K_{\mathcal{A}})$ is an isomorphism, and hence $S_{\mathcal{A}}$ is full and faithful by (the dual of) [14, IV.3, Thm. 1]. Similar arguments show that the functor $S_{\mathcal{B}}$ has the same properties as those just established for $S_{\mathcal{A}}$. Therefore, the functors $S_{\mathcal{A}}$ and $S_{\mathcal{B}}$ meet the requirements in Proposition 3.6.

It remains to verify conditions (a)–(c) in Theorem 3.7. Condition (a) is straightforward from Definition 5.2, and (b) holds by Lemma 5.3. For every object $T = (X,Y,\theta)$ in $(U\!\downarrow\! V)$ the counit $S_{\mathcal{A}}K_{\mathcal{A}}T \to T$ is monic as $\operatorname{Ker}(U^!\theta \circ \eta_X) \rightarrowtail X$ and $0 \rightarrowtail Y$ are monics in $\mathcal{A}$ and $\mathcal{B}$. The counit $S_{\mathcal{B}}K_{\mathcal{B}}T \to T$ is monic as $0 \rightarrowtail X$ and $Y \rightarrowtail Y$ are monics. Hence (c) holds. □

5.4 Example. Let A and B be rings and let $M = {}_BM_A$ be a (B,A)-bimodule. We consider the comma category associated to $U = M \otimes_A -\colon A\text{-Mod} \to B\text{-Mod}$ and V being the identity functor on B-Mod. Theorem B yields a homeomorphism and an order-isomorphism,

$$f\colon \operatorname{ASpec}(A\text{-Mod}) \sqcup \operatorname{ASpec}(B\text{-Mod}) \longrightarrow \operatorname{ASpec}\left((M \otimes_A -)\!\downarrow\!(B\text{-Mod})\right),$$

which we now describe in more detail. There is a well-known equivalence of categories,

$$((M \otimes_A -)\!\downarrow\!(B\text{-Mod})) \xrightarrow{\;E\;} T\text{-Mod} \qquad \text{where} \qquad T = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix};$$

see [5] and [7, Thm. (0.2)]. Under this equivalence, an object (X,Y,θ) in the comma category is mapped to the left T-module whose underlying abelian group is $X \oplus Y$ where T-multiplication is defined by

$$\begin{pmatrix} a & 0 \\ m & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ \theta(m \otimes x) + by \end{pmatrix} \qquad \text{for} \qquad \begin{pmatrix} a & 0 \\ m & b \end{pmatrix} \in T \quad \text{and} \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \begin{matrix} X \\ \oplus \\ Y \end{matrix}\,.$$

For simplicity we now consider the case where A and B are commutative (but T is not). Define a map $\tilde{f}$ by commutativity of the diagram

$$\begin{array}{ccc} \operatorname{Spec} A \sqcup \operatorname{Spec} B & \overset{\tilde{f}}{\dashrightarrow} & \operatorname{ASpec}(T\text{-Mod}) \\ {\scriptstyle q_A \sqcup q_B}\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\uparrow{\scriptstyle \operatorname{ASpec} E} \\ \operatorname{ASpec}(A\text{-Mod}) \sqcup \operatorname{ASpec}(B\text{-Mod}) & \xrightarrow{\ f\ } & \operatorname{ASpec}\left((M \otimes_A -) \downarrow (B\text{-Mod})\right), \end{array}$$

where q_A and q_B are the homeomorphisms and order-isomorphisms from 2.5. By using the definitions of these maps, it follows easily that

$$\tilde{f}(\mathfrak{p}) = \left\langle T \Big/ \begin{pmatrix} \mathfrak{p} & 0 \\ M & B \end{pmatrix} \right\rangle \qquad \text{and} \qquad \tilde{f}(\mathfrak{q}) = \left\langle T \Big/ \begin{pmatrix} A & 0 \\ M & \mathfrak{q} \end{pmatrix} \right\rangle$$

for $\mathfrak{p} \in \operatorname{Spec} A$ and $\mathfrak{q} \in \operatorname{Spec} B$. In the terminology of [10, Def. 6.1] the denominators above are *comonoform* left ideals in T. For $A = B = M = K$, a field, this recovers [10, Exa. 8.3][‡]. For $A = B = M = \Bbbk$, where $\Bbbk$ is any commutative ring, the conclusion above also follows from Example 4.13 with $n = 2$.

Appendix A. Quivers with relations and their representations

In this appendix, we present some (more or less standard) background material on representations of quivers with relations that we will need, and take for granted, in Section 4.

A.1. A *quiver* is a directed graph. For a quiver Q we denote by Q_0 the set of vertices and by Q_1 the set of arrows in Q. Unless otherwise specified there are no restrictions on a quiver; it may have infinitely many vertices, it may have loops and/or oriented cycles, and there may be infinitely many or no arrows from one vertex to another.

For an arrow $a\colon i \to j$ in Q the vertex i, respectively, j, is called the *source*, respectively, *target*, of a. A *path* p in Q is a finite sequence of composable arrows $\bullet \xrightarrow{a_1} \bullet \xrightarrow{a_2} \cdots \xrightarrow{a_n} \bullet$ (that is, the target of a_ℓ equals the source of $a_{\ell+1}$), which we write $p = a_n \cdots a_2 a_1$. If p and q are paths in Q and the target of q coincides with the source of p, then we write pq for the composite path (i.e. first q, then p). At each vertex $i \in Q_0$ there is by definition a *trivial path*, denoted by e_i, whose source and target are both i. For every path p in Q with source i and target j one has $pe_i = p = e_j p$.

Let Q be a quiver and let $\mathcal{A}$ be an abelian category. One can view Q as a category, which we denote by $\bar{Q}$, whose objects are vertices in Q and whose morphisms are paths in Q. An *$\mathcal{A}$-valued representation of Q* is a functor $X\colon \bar{Q} \to \mathcal{A}$ and a morphism $\lambda\colon X \to Y$ of representations X and Y is a natural transformation. The category of $\mathcal{A}$-valued representations of Q, that is, the category of functors $\bar{Q} \to \mathcal{A}$, is written $\operatorname{Rep}(Q, \mathcal{A})$. In symbols:

$$\operatorname{Rep}(Q, \mathcal{A}) = \operatorname{Func}(\bar{Q}, \mathcal{A}). \tag{$\sharp$6}$$

It is an abelian category where kernels and cokernels are computed vertexwise.

A.2. Let $\Bbbk$ be a commutative ring. Recall that a *$\Bbbk$-linear category* is a category $\mathcal{K}$ enriched in the monoidal category $\Bbbk$-Mod of $\Bbbk$-modules, that is, the hom-sets in $\mathcal{K}$ have structures of $\Bbbk$-modules and composition in $\mathcal{K}$ is $\Bbbk$-bilinear. If $\mathcal{K}$ and $\mathcal{L}$ are $\Bbbk$-linear categories, then we write $\operatorname{Func}_\Bbbk(\mathcal{K}, \mathcal{L})$ for the category of $\Bbbk$-linear functors from $\mathcal{K}$ to $\mathcal{L}$. Here we must require that $\mathcal{K}$ is skeletally small in order for $\operatorname{Func}_\Bbbk(\mathcal{K}, \mathcal{L})$ to have small hom-sets.

If $\mathcal{C}$ is any category we write $\Bbbk\mathcal{C}$ for the category whose objects are the same as those in $\mathcal{C}$ and where $\operatorname{Hom}_{\Bbbk\mathcal{C}}(X,Y)$ is the free $\Bbbk$-module on the set $\operatorname{Hom}_{\mathcal{C}}(X,Y)$. Composition in $\Bbbk\mathcal{C}$ is induced by composition in $\mathcal{C}$. The category $\Bbbk\mathcal{C}$ is evidently $\Bbbk$-linear and we call it the

[‡] This example, which inspired the present paper, was worked out using methods different from what we have developed here. The approach in [10, Exa. 8.3] is that one can write down *all* ideals in a lower triangular matrix ring, see for example [13, Prop. (1.17)], and from this list it is possible to single out the comonoform ones.

$\Bbbk$-*linearization* of $\mathcal{C}$. Note that there is canonical functor $\mathcal{C} \to \Bbbk\mathcal{C}$. For any skeletally small category $\mathcal{C}$ and any $\Bbbk$-linear category $\mathcal{L}$ there is an equivalence of categories,

$$\mathrm{Func}(\mathcal{C},\mathcal{L}) \simeq \mathrm{Func}_{\Bbbk}(\Bbbk\mathcal{C},\mathcal{L})\,. \qquad (\sharp 7)$$

That is, (ordinary) functors $\mathcal{C} \to \mathcal{L}$ corrspond to $\Bbbk$-linear functors $\Bbbk\mathcal{C} \to \mathcal{L}$. This equivalence maps a functor $F\colon \mathcal{C} \to \mathcal{L}$ to the $\Bbbk$-linear functor $\tilde{F}\colon \Bbbk\mathcal{C} \to \mathcal{L}$ given by $\tilde{F}(C) = F(C)$ for any object C and $\tilde{F}(x_1\varphi_1 + \cdots + x_m\varphi_m) = x_1F(\varphi_1) + \cdots + x_mF(\varphi_m)$ for any morphism $x_1\varphi_1 + \cdots + x_m\varphi_m$ in $\Bbbk\mathcal{C}$ (where $x_u \in \Bbbk$ and $\varphi_1,\dots,\varphi_m\colon C \to C'$ are morphisms in $\mathcal{C}$). In the other direction, ($\sharp$7) maps a $\Bbbk$-linear functor $\Bbbk\mathcal{C} \to \mathcal{L}$ to the composition $\mathcal{C} \to \Bbbk\mathcal{C} \to \mathcal{L}$.

A *two-sided ideal* $\mathcal{I}$ in a $\Bbbk$-linear category $\mathcal{K}$ is a collection of $\Bbbk$-submodules $\mathcal{I}(X,Y) \subseteq \mathrm{Hom}_{\mathcal{K}}(X,Y)$, indexed by pairs (X,Y) of objects in $\mathcal{K}$, such that

- For every $\beta \in \mathrm{Hom}_{\mathcal{K}}(Y,Y')$ and $\varphi \in \mathcal{I}(X,Y)$ one has $\beta\varphi \in \mathcal{I}(X,Y')$, and
- For every $\varphi \in \mathcal{I}(X,Y)$ and $\alpha \in \mathrm{Hom}_{\mathcal{K}}(X',X)$ one has $\varphi\alpha \in \mathcal{I}(X',Y)$.

Given such an ideal $\mathcal{I}$ in $\mathcal{K}$ one can define the quotient category $\mathcal{K}/\mathcal{I}$, which has the same objects as $\mathcal{K}$ and hom-sets defined by (quotient of $\Bbbk$-modules):

$$\mathrm{Hom}_{\mathcal{K}/\mathcal{I}}(X,Y) = \mathrm{Hom}_{\mathcal{K}}(X,Y)/\mathcal{I}(X,Y)\,.$$

Composition in $\mathcal{K}/\mathcal{I}$ is induced from composition in $\mathcal{K}$, and it is well-defined since $\mathcal{I}$ is a two-sided ideal. It is straightforward to verify the $\mathcal{K}/\mathcal{I}$ is a $\Bbbk$-linear category. There is a canonical $\Bbbk$-linear functor $\mathcal{K} \to \mathcal{K}/\mathcal{I}$, which for any $\Bbbk$-linear category $\mathcal{L}$ induces a functor $\mathrm{Func}_{\Bbbk}(\mathcal{K}/\mathcal{I},\mathcal{L}) \to \mathrm{Func}_{\Bbbk}(\mathcal{K},\mathcal{L})$. It is not hard to see that this functor is fully faithful, so $\mathrm{Func}_{\Bbbk}(\mathcal{K}/\mathcal{I},\mathcal{L})$ may be identified with a full subcategory of $\mathrm{Func}_{\Bbbk}(\mathcal{K},\mathcal{L})$. In fact, one has

$$\mathrm{Func}_{\Bbbk}(\mathcal{K}/\mathcal{I},\mathcal{L}) \simeq \{F \in \mathrm{Func}_{\Bbbk}(\mathcal{K},\mathcal{L}) \mid F \text{ kills } \mathcal{I}\}\,.$$

If $\mathcal{R}$ is a collection of morphisms in a $\Bbbk$-linear category $\mathcal{K}$, then we write $(\mathcal{R})$ for the two-sided ideal in $\mathcal{K}$ generated by $\mathcal{R}$. I.e. $(\mathcal{R})(X,Y)$ consists of finite sums $\sum_u x_u\beta_u\varphi_u\alpha_u$ where $x_u \in \Bbbk$ and $\alpha_u\colon X \to X_u$, $\varphi_u\colon X_u \to Y_u$, $\beta_u\colon Y_u \to Y$ are morphisms in $\mathcal{K}$ with $\varphi_u \in \mathcal{R}$.

A.3. Let Q be a quiver and let $\Bbbk$ be a commutative ring. Consider the $\Bbbk$-linear category $\Bbbk\bar{Q}$, that is, the $\Bbbk$-linearization (see A.2) of the category $\bar{Q}$ (see A.1).

A *relation* (more precisely, a $\Bbbk$-*linear relation*) in Q is a morphism ρ in $\Bbbk\bar{Q}$. That is, ρ is a formal $\Bbbk$-linear combination $\rho = x_1p_1 + \cdots + x_mp_m$ ($x_u \in \Bbbk$) of paths $p_1,\dots,p_m$ in Q with a common source and a common taget.

A *quiver with relations* is a pair $(Q,\mathcal{R})$ with Q a quiver and $\mathcal{R}$ a set of relations in Q.

Let $\mathcal{A}$ be a $\Bbbk$-linear abelian category. For a representation $X \in \mathrm{Rep}(Q,\mathcal{A})$, as in A.1, and a relation $\rho = x_1p_1 + \cdots + x_mp_m$ in Q, define $X(\rho) := x_1X(p_1) + \cdots + x_mX(p_m)$. One says that X *satisfies* the relation ρ if $X(\rho) = 0$.

If $(Q,\mathcal{R})$ is a quiver with relations, then an $\mathcal{A}$-*valued representation of* $(Q,\mathcal{R})$ is a representation $X \in \mathrm{Rep}(Q,\mathcal{A})$ with $X(\rho) = 0$ for all $\rho \in \mathcal{R}$, that is, X satisfies all relations in $\mathcal{R}$. We write $\mathrm{Rep}((Q,\mathcal{R}),\mathcal{A})$ category of $\mathcal{A}$-valued representations of $(Q,\mathcal{R})$. In symbols:

$$\mathrm{Rep}((Q,\mathcal{R}),\mathcal{A}) = \{X \in \mathrm{Rep}(Q,\mathcal{A}) \mid X(\rho) = 0 \text{ for all } \rho \in \mathcal{R}\}\,.$$

We consider $\mathrm{Rep}((Q,\mathcal{R}),\mathcal{A})$ as a full subcategory of $\mathrm{Rep}(Q,\mathcal{A})$. We have a diagram:

$$\begin{array}{ccc}
\mathrm{Rep}(Q,\mathcal{A}) & \xrightarrow{\ \simeq\ } & \mathrm{Func}_{\Bbbk}(\Bbbk\bar{Q},\mathcal{A}) \\
\uparrow & & \uparrow \\
\mathrm{Rep}((Q,\mathcal{R}),\mathcal{A}) & \xrightarrow{\ \simeq\ } & \mathrm{Func}_{\Bbbk}(\Bbbk\bar{Q}/(\mathcal{R}),\mathcal{A})\,,
\end{array}$$

where the upper horizontal equivalence comes from ($\sharp$6) and ($\sharp$7). The vertical functors are inclusions. It is immediate from the definitions that the equivalence in the top row restricts to an equivalence in the bottom row, so we get commutativity of the displayed diagram.

A.4. Let Q be a quiver with finitely many vertices(!) and let $\Bbbk$ be a commutative ring. The *path algebra* $\Bbbk Q$ is the $\Bbbk$-algebra whose underlying $\Bbbk$-module is free with basis all paths in Q and multiplication of paths p and q are given by their composition pq, as in A.1, if they are composable, and $pq = 0$ if they are not composable. Note that $\Bbbk Q$ has unit $\sum_{i \in Q_0} e_i$.

There is an equivalence of categories, see e.g. [2, Lem. p. 6] or [1, Chap. III.1 Thm. 1.6]:

$$\operatorname{Rep}(Q, \Bbbk\text{-Mod}) \simeq \Bbbk Q\text{-Mod}\,. \tag{$\sharp$8}$$

We describe the quasi-inverse functors U and V that give this equivalence. A representation X is mapped to the left $\Bbbk Q$-module UX whose underlying $\Bbbk$-module is $\bigoplus_{i \in Q_0} X(i)$; multiplication by paths works as follows: Let $\varepsilon_i \colon X(i) \rightarrowtail \bigoplus_{i \in Q_0} X(i)$ and $\pi_i \colon \bigoplus_{i \in Q_0} X(i) \twoheadrightarrow X(i)$ be the i^{th} injection and projection in $\Bbbk$-Mod. For a path $p \colon i \rightsquigarrow j$ and an element $z \in UX$ one has $pz = (\varepsilon_j \circ X(p) \circ \pi_i)(z)$. In the other direction, a left $\Bbbk Q$-module M is mapped to the representation VM given by $(VM)(i) = e_i M$ for $i \in Q_0$. For a path $p \colon i \rightsquigarrow j$ in Q the $\Bbbk$-homomorphism $(VM)(p) \colon e_i M \to e_j M$ is left multiplication by p.

By definition, see A.3, a relation in Q can be viewed as an element (of a special kind) in the algebra $\Bbbk Q$. If $(Q, \mathcal{R})$ is a quiver with relations and $I = (\mathcal{R})$ is the two-sided ideal in $\Bbbk Q$ generated by the subset $\mathcal{R} \subseteq \Bbbk Q$, then we have a diagram:

$$\begin{array}{ccc} \operatorname{Rep}(Q, \Bbbk\text{-Mod}) & \xrightarrow{\ \simeq\ } & \Bbbk Q\text{-Mod} \\ \uparrow & & \uparrow \\ \operatorname{Rep}((Q, \mathcal{R}), \Bbbk\text{-Mod}) & \xrightarrow{\ \simeq\ } & \Bbbk Q/I\text{-Mod}\,, \end{array}$$

where the upper horizontal equivalence is ($\sharp$8). The vertical functors are inclusions, where $\Bbbk Q/I$-Mod is identified with the full subcategory $\{M \in \Bbbk Q\text{-Mod} \mid IM = 0\}$ of $\Bbbk Q$-Mod. It is immediate from the definitions that the equivalence in the top row restricts to an equivalence in the bottom row, so we get commutativity of the displayed diagram.

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